Published by Institute of Physics Publishing for SISSA/ISAS
Received: *July 27, 2004*
Revised: *September 13, 2004*
Accepted: *September 14, 2004*

# Lattice supersymmetry, superfields and renormalization

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Abstract: We study euclidean lattice formulations of non-gauge supersymmetric models with up to four supercharges in various dimensions. We formulate the conditions under which the interacting lattice theory can exactly preserve one or more nilpotent anticommuting supersymmetries. We introduce a superfield formalism, which allows the enumeration of all possible lattice supersymmetry invariants. We use it to discuss the formulation of Q-exact lattice actions and their renormalization in a general manner. In some examples, one exact supersymmetry guarantees finiteness of the continuum limit of the lattice theory. As a consequence, we show that the desired quantum continuum limit is obtained without fine tuning for these models. Finally, we discuss the implications and possible further applications of our results to the study of gauge and non-gauge models.

Keywords: Field Theories in Lower Dimensions, Sigma Models, Lattice Quantum Field Theory, Extended Supersymmetry.

## Contents



## 1. Introduction

### 1.1 Motivation

Supersymmetric field theories enjoy remarkable perturbative nonrenormalization properties, as was first noticed in the 4d Wess-Zumino models at 1-loop [1] and then to all orders [2]. Remarkably, a nonperturbative nonrenormalization theorem was proven for the 4d Wess-Zumino model with a cubic superpotential interaction [3]. On the other hand, in the more interesting and phenomenologically relevant case of super-QCD, it is known that the tree level superpotential does receive nonperturbative corrections [4]. Indeed, nonperturbative contributions to the superpotential are often invoked in scenarios for moduli stabilization and spontaneous supersymmetry breaking in a variety of supersymmetric extensions to the Standard Model of particle physics.



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The above examples illustrate the importance of reliable nonperturbative methods of analysis in supersymmetric field theories. While traditional methods based on holomorphy and symmetry, including embedding the theories in various string constructions, have resulted in stunning progress for many theories, it would be advantageous to have a comparable wealth of results from other methods. The most pressing reason is that we would like to answer questions that the traditional methods are not able to address, such as nonperturbative contributions to nonholomorphic quantities. One technique for nonperturbative analysis which is worth exploring is lattice regularization. It is this particular approach that we study here. Moreover, the lattice is the only known fundamental, nonperturbative definition of a general quantum field theory; as a matter of principle, we believe that it is important to have such a definition for supersymmetric field theories.

Typically in field theory, one seeks a regulator that preserves all of the symmetries present at tree level. Otherwise, symmetry breaking will be induced, producing spurious results. In the continuum formulation of supersymmetric theories, various methods which meet this requirement have been developed over the years: a supersymmetric Pauli-Villars sector, dimensional reduction, supersymmetric higher derivative terms, etc. However, in 30 years no fully supersymmetric lattice regulator of an interacting supersymmetric field theory has been constructed. The best that has been so far achieved are lattice models where it can be argued that the target supersymmetric field theory is obtained with little or no fine-tuning in the quantum continuum limit. In these cases, the full supersymmetry is recovered in one of two ways. First, without fine-tuning, as an accidental symmetry that results from symmetries of the lattice theory [5]–[6]; in the case of 4d $\mathcal{N} = 1$ super-Yang-Mills, chiral symmetry guarantees continuum supersymmetry, either through a modest fine tuning with Wilson fermions [7] or, without fine-tuning, through chiral lattice fermions [8]. The second case is where supersymmetry is recovered due to finiteness[1] or super-renormalizability [9]–[18]. In the constructions considered below, we will see aspects of both these cases.

### 1.2 The problem

The main difficulty for preserving exact supersymmetry on the lattice is to respect the supersymmetry of the interaction terms.[2] The main obstacle to achieving this is the failure of the Leibnitz rule for lattice derivatives.[3] To elucidate, recall that supersymmetry generators can be represented as differential operators acting on functions of "superspace" $(x, \theta, \dots \theta')$. A supersymmetry generator $Q$ typically takes the form:

$$Q = \frac{\partial}{\partial \theta} + \theta \Gamma \frac{\partial}{\partial x}\,, \tag{1.1}$$

[1] Here, and in the remainder of this article, "finite" will have the following technical meaning: the sum of diagrams contributing to any proper vertex has UV degree $D < 0$ according to the rules of power-counting in lattice perturbation theory. A specific example will be discussed in appendix D.

[2] For a lagrangian quadratic in the fields, it is possible to exactly preserve all supersymmetry on the lattice, given a judicious choice of lattice derivatives; see section 2 and appendix B. We are not interested in this trivial case, and do not enumerate the many articles that have focused solely on such constructions.

[3] Not, as sometimes stated, the fermion doubling problem; it would only be an obstacle for chiral supersymmetric theories.

where we omitted various indices (i.e., the details of the constant matrices $\Gamma$, indices of $\theta$, etc.) as inessential for the general argument here. A supersymmetric action is then given as an integral over superspace of a local function of superfields $\Phi(x, \theta, \ldots \theta')$:

$$S = \int dx d\theta \cdots d\theta' \, F(\Phi) \,. \tag{1.2}$$

Supersymmetry is generated by the $Q$ action on the superfields $\delta_\epsilon \Phi = \epsilon Q \Phi$:

$$\begin{aligned} \delta_\epsilon S &= \int dx d\theta \cdots d\theta' [F(\Phi + \epsilon Q \Phi) - F(\Phi))] \\ &= \int dx d\theta \cdots d\theta' \, \epsilon Q F(\Phi) \\ &= \int dx d\theta \cdots d\theta' \, \epsilon \left( \frac{\partial}{\partial \theta} + \theta \Gamma \frac{\partial}{\partial x} \right) F(\Phi) = 0 \,. \end{aligned} \tag{1.3}$$

The two terms on the last line vanish separately, for somewhat different reasons: the first term is zero because the $\partial / \partial \theta$ derivative eliminates the corresponding $\theta$ from the lagrangian and the remaining $\int d\theta$ is zero due to Grassmann integration rules.

The vanishing of the second term — a total derivative and hence a surface term — in the last line of (1.3) is more interesting (from our present perspective). In going from the first to the second line of (1.3), we asserted that $F(\Phi + \epsilon Q \Phi) - F(\Phi) = \epsilon Q F(\Phi)$ and implicitly used the Leibnitz rule for spacetime derivatives. On the lattice, however, spacetime derivatives are replaced by finite differences, for which the Leibnitz rule fails. A naive latticization of a supersymmetric action will then have a supersymmetry variation that is a lattice total derivative (whose contribution to the variation vanishes, as in (1.3)) plus corrections of order the lattice spacing, which spoil the supersymmetry of the action — a fact that was originally pointed out by Dondi and Nicolai [19]. Thus, quite generally, a naive latticization of an interacting supersymmetric lagrangian will lead to a nonsupersymmetric lattice action, whose supersymmetry variation is proportional to powers of the lattice spacing.

In the naive (i.e., classical) continuum limit, supersymmetry is, of course, restored. However, quantum effects may generate a number of relevant operators violating the continuum limit supersymmetry — for example, soft masses for scalars, which are not forbidden by any symmetry (but supersymmetry). These then require a possibly large number of fine tunings to recover the supersymmetric continuum limit. The need for fine tuning, while theoretically palatable, renders the lattice studies of interesting nonperturbative phenomena in supersymmetric theories, such as dynamical (super-) symmetry breaking practically impossible.

### 1.3 The approach

The natural question to ask, then, is whether it is possible to preserve at least a subset of the supersymmetry transformations in the interacting lattice theory. As shown above, the source of the difficulty is the second term in $Q$ of eq. (1.1), proportional to space-time derivatives. If a supercharge could be represented, upon conjugation (a change of basis) in a form involving only $\theta$-derivatives, the Leibnitz rule for spacetime derivatives would not be needed to ensure supersymmetric invariance of the action. Thus, the corresponding supersymmetries stand the chance to be exact symmetries of the lattice interactions. It is clear — and we will see many examples — that nilpotent anticommuting supercharges can always be conjugated to pure $\theta$-derivatives. We will explain how a latticization of the continuum action that preserves these nilpotent anticommuting supersymmetries can be performed in most cases.

However, the existence of nilpotent supercharges is not sufficient to guarantee that all interactions on the lattice are invariant under the corresponding supersymmetries. This will be explained in more detail in section 4; here we only state our general conclusions in this regard:

1. If the continuum supersymmetry algebra contains one or more nilpotent anticommuting supercharges, supersymmetric actions that are integrals over full superspace (e.g., D-terms) can be latticized while preserving all these supercharges.

2. Continuum interaction terms given by integrals over parts of the full superspace (e.g., over chiral superspace, F-terms) remain invariant after latticization only if there exists a (linear combination of) nilpotent supercharge(s) such that the supersymmetry variation of these interactions terms is not a total derivative.

The above criteria 1.) and 2.) are necessary to define a supersymmetric lattice partition function. However, they are not sufficient to ensure that it will have the desired continuum limit, even classically. As we will see, while studying some 2d $(2,2)$ theories, the removal of fermion doublers with the simultaneous preservation of the exact supersymmetry of the partition function and of other desirable symmetries of the continuum theory is not always possible within the present approach.

Nevertheless, in many cases we will be able to write down classical lattice actions that preserve part of the continuum supersymmetry and have the desired continuum limit spectrum and interactions. One can hope, then, that the exact lattice supersymmetry, perhaps combined with other symmetries of the lattice action, precludes the generation at the quantum level of relevant operators that break the continuum limit supersymmetry. Thus, the goal of this approach to lattice supersymmetry is to find examples of interacting lattice theories where reaching the supersymmetric critical point requires no fine tuning (or, at least, where the number of required fine-tunings is reduced, compared to a naive discretization). Studying this question is the main thrust of this paper.

In fact, all current euclidean lattice formulations of interacting supersymmetric theories that maintain part of the continuum supersymmetry algebra do so by preserving precisely such a nilpotent subset of the supersymmetry transformations.[4] The earliest example of a theory preserving part of the continuum limit supersymmetry is the spatial lattice

[4]The hamiltonian methods using super-DLCQ, preserve the entire supersymmetry algebra; for a review, see [20].



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approach of refs. [9]–[12], yielding a lattice hamiltonian (invariant under supersymmetries whose anticommutator does not include lattice translations); there have been numerous studies of this approach since, a recent example being [21].

More recently, there has been a revival of the study of theories with supersymmetry on both spatial and euclidean lattices, coming from two different directions. The first is the exploitation of relations to topological quantum field theory [16], closely related to the earlier work of [12, 10]. The second direction is the construction of supersymmetric noncompact lattice gauge theories using a "top-down" approach, based on the orbifolding of "mother" models, a.k.a. "deconstruction" [5]. Finally, ref. [6] recently made an interesting proposal to latticize compact supersymmetric gauge theories, bearing many features in common with the first of the above approaches. In all these models, the exact lattice supersymmetries derive from one or more nilpotent anticommuting supercharges.

We pause to mention a few important technical points. There are cases where the exact lattice supercharges can not be conjugated to pure-$\theta$ derivatives. Examples exist both in gauge and non-gauge models with lattice supersymmetry. A non-gauge example is given in this paper in our study of the 2d $(2,2)$ theories with twisted mass term, see eq. (C.7), where the two exact nilpotent lattice supercharges do not anticommute — the anticommutator of the two nilpotent supercharges is proportional to the central charge. In the case of gauge theories (not the topic of this paper), if we insist on avoiding the introduction of unphysical degrees of freedom, we also expect to have exact lattice supersymmetries that anticommute or are nilpotent only up to gauge transformations. This is because the anticommutator of two supercharges in Wess-Zumino "gauge" involves gauge transformations, in addition to spacetime translations. Appropriate examples are described in the recent literature: in the "deconstruction" lattice models with eight supercharges of the target theory [5], anticommutativity of the two exact lattice supercharges holds only up to gauge transformations, while in the construction of [6] it is the nilpotency of the exact supercharge that holds only up to gauge transformations. Finally, we also note that while all lattice supersymmetric actions in this paper are Q-exact, it can happen that terms appear in the action that are only Q-closed; an example of this can be found in the 3d lattice of [5].

We summarize this section by stressing the unifying theme of all formulations of lattice supersymmetry mentioned above: the lattice theory exactly preserves some anticommuting nilpotent (perhaps up to gauge transformations) supersymmetries. The rest of the supersymmetries are approximate and it is hoped or argued that they are restored in the quantum continuum limit. In this paper, we begin to develop this approach to lattice supersymmetry in a systematic "bottom-up" manner and to explore the possibilities of constructing supersymmetric lattice actions in different models with up to four supercharges in various dimensions.

### 1.4 Summary

Many of the supersymmetric lattice actions found in this paper are not new; however, as we will explain in the main text, they have been written using methods different from ours [12, 16], which somewhat obscure the study of symmetries and renormalization. We develop a unified formalism, which allows for the straightforward construction of all possible terms



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invariant under the lattice supersymmetry. The ability to write down all invariants under the lattice (super)symmetries is especially helpful to study the possible supersymmetric deformations of the lattice actions, their renormalization, the approach to the continuum limit, and the corresponding restoration of (super)symmetries. In addition, we hope that our approach can be generalized to gauge models.

We begin, in section 2, by describing the simplest example of a lattice system with one exact supersymmetry: quantum mechanics on a euclidean "supersymmetric" lattice. All technical elements that are important in higher-dimensional examples are present here in their simplest form, hence we give sufficient detail. The result is a lattice partition function, eqs. (2.18) and (2.19), preserving one exact supersymmetry. For this 1d lattice, the full supersymmetry is automatically restored in the continuum limit. In a forthcoming paper, we will describe why this is not true for, say, the naive discretization. (Monte Carlo simulution results presented in [13] have already shown an indication of this.) To hint at the results which will be shown there, a diagram with UV degree $D = 0$ in the lattice perturbation theory gives a finite contribution — coming from doubler modes associated with the Wilson fermions — that is not present in the continuum perturbation theory. This effect is avoided in the present approach, due to cancellations that occur due to the exact lattice supersymmetry. Besides this vital feature, it is worth noting that the supersymmetric partition function enjoys a number of desirable properties: we show that the Witten index can be exactly calculated already at finite lattice spacing; furthermore, numerical analysis has shown that the approach to the supersymmetric continuum limit is faster and exhibits boson-fermion spectrum degeneracy [13, 14]. Much of the detail supporting these statements is given in appendix A.

We continue, by increasing the dimensionality of spacetime, in sections 3 and 4, where we study the applicability of criteria 1.) and 2.) from section 1.3 to different 2d and 3d theories with two supercharges, and to 3d and 4d theories with four supercharges. The first set of examples, considered in section 3, are the two-supercharge 3d and the $(1,1)$ 2d theories. These have no nilpotent supercharge that can be preserved by the interactions; thus criterion 1.) is already violated. The second example, considered in section 4, is the $(2,0)$ 2d theory. It illustrates the necessity of criterion 2.). This provides an example of an important class of theories, where not all interaction terms can preserve the nilpotent supersymmetry. Many interesting theories, such as the 4d $\mathcal{N} = 1$ Wess-Zumino model and its 3d compactification, fall into this class.

The main body of this paper, section 5, is devoted to the study of 2d $(2,2)$ models with chiral superfields. We begin, in section 5.1, by examining the two possible choices, "$A$"-type and "$B$"-type, of pairs of supercharges to be preserved by the lattice and explain why we focus on the $A$-type charges. We then introduce lattice superfields and the action of the lattice counterparts of the continuum limit global symmetries. Much of the technical detail of the construction of lattice superfields is given in appendix B.

In section 5.2 we show that the most general lattice action, preserving the $A$-type supercharges and the maximal number of global symmetries, is a lattice version of the $(2,2)$ sigma model. While this action has the correct naive continuum limit, it suffers a fermion doubling problem. We discuss two possible approaches to the doubling problem, in appendix C and in section 5.3. The first allows the doublers to be lifted at the expense of breaking all the lattice supersymmetry, while the second preserves one nilpotent supercharge.

In appendix C, we study the twisted mass version of the sigma model and show that twisted nonlocal mass terms can be used to lift the doublers at the expense of breaking all the lattice supersymmetry. Since generic 2d supersymmetric sigma models are renormalizable (rather than superrenormalizable), a lattice implementation of the models with twisted Wilson terms may require fine tuning in the quantum continuum limit.[5]

In section 5.3, we find the general formulae for lattice superpotential $F$-terms and show that the doublers can be lifted via $F$-type nonlocal mass terms, while preserving a single nilpotent supercharge. We show that the most general supersymmetric lattice action, consistent with the symmetries, admits non-holomorphic and Lorentz violating relevant terms (which are related to each other by the exact supersymmetry). Thus, the supersymmetric lattice action with the desired $(2, 2)$ continuum limit is not generic.

We then consider renormalization of lattice models whose classical continuum limits are 2d $(2, 2)$ models with good ultraviolet behavior. In appendix D, we study Wess-Zumino (or Landau-Ginzburg) models that are ultraviolet finite and show that the one exact lattice supersymmetry is sufficient to guarantee finiteness of the lattice theory in the zero lattice spacing limit. This then allows us, in section 5.3, to argue that the non-holomorphic and Lorentz violating terms are irrelevant quantum mechanically and that the $(2, 2)$ continuum limit is achieved without fine tuning. Further general properties of the supersymmetric lattice version of the Wess-Zumino model, such as positivity of the fermion determinant (for square lattices) are also shown in appendix D.

### 1.5 Outlook

The examples considered in this paper show that euclidean actions of supersymmetric models, obeying the criteria of section 1.3, can be latticized with the simultaneous preservation of a number of exact nilpotent supersymmetries, while the rest of the supersymmetries are respected only up to (strictly positive) powers of the lattice spacing.

To find whether the exact lattice supersymmetry guarantees that the supersymmetric and Lorentz invariant quantum continuum limit is achieved without fine tuning requires further analysis. In the interesting example of general 2d $(2, 2)$ models, we showed that the supersymmetric lattice action with the desired continuum limit is not generic. In other words, obtaining the $(2, 2)$ quantum continuum limit either involves fine-tuning or requires favorable ultraviolet properties of the target continuum theories. The 2d $(2, 2)$ finite examples, that are shown here to not require fine-tuning, indicate that in 3d and 4d this program is more likely to succeed in theories with a large number of supersymmetries, since it is these that are typically ultraviolet finite.

Thus, the scope of the present approach to lattice supersymmetry, while limited, includes other interesting models, whose latticization along the lines presented here is worth

[5]It may still be possible that for particular classes of sigma models the fine tuning is minimal or exhausted at one loop; we leave this for future study.

studying. Some obvious types of theories are missing from our analysis, notably, those with more supersymmetry, and particularly, gauge theories. It would be interesting to make explicit contact between our approach and the recent proposal of Sugino. This would allow us to investigate, along the lines of this paper, the possible deformations and renormalization of the proposed supersymmetric $(2,2)$ lattice gauge theory actions.

The results of section 5.3 below (as well as of appendix D) allow us to conclude that the $(2,2)$ LG models can be simulated on the lattice. Depending on the superpotential, these models have infrared fixed points described by the $\mathcal{N}=2$ minimal models of conformal field theory (see, e.g., the second reference in [23]). It would be of some interest, at the very least as a nonperturbative check of the lattice techniques, to numerically verify the predicted values of critical exponents. Work is in progress along these lines, which we intend to report upon in the near future.

## 2. Supersymmetric quantum mechanics on a supersymmetric lattice

In this section, we consider supersymmetric quantum mechanics on a supersymmetric lattice. This is a simple enough example that illustrates all major techniques we use to construct supersymmetric lattice actions. Thus, in this section, we give sufficient detail. Since the details of the construction of higher dimensional theories are more notationally involved, many are given in the appendices.

The supersymmetry algebra is generated by two real nilpotent supercharges, with $\{Q_1, Q_2\} = 2i\frac{\partial}{\partial t}$. The supercharges can be represented in terms of differential operators acting on functions of time $t$ and two real anticommuting variables $(\theta^1, \theta^2)$:

$$\begin{aligned} Q_1 &= \frac{\partial}{\partial \theta^1} + i\,\theta^2\,\frac{\partial}{\partial t} = e^{-i\theta^1\theta^2\frac{\partial}{\partial t}}\,\frac{\partial}{\partial\theta^1}\,e^{i\theta^1\theta^2\frac{\partial}{\partial t}} \\ Q_2 &= \frac{\partial}{\partial \theta^2} + i\,\theta^1\,\frac{\partial}{\partial t} = e^{i\theta^1\theta^2\frac{\partial}{\partial t}}\,\frac{\partial}{\partial\theta^2}\,e^{-i\theta^1\theta^2\frac{\partial}{\partial t}}\,. \end{aligned} \tag{2.1}$$

The supercovariant derivatives $D_1, D_2$ anticommute with $Q_1, Q_2$ and are also represented as differential operators:

$$\begin{aligned} D_1 &= e^{i\theta^1\theta^2\frac{\partial}{\partial t}}\,\frac{\partial}{\partial\theta^1}\,e^{-i\theta^1\theta^2\frac{\partial}{\partial t}} \\ D_2 &= e^{-i\theta^1\theta^2\frac{\partial}{\partial t}}\,\frac{\partial}{\partial\theta^2}\,e^{i\theta^1\theta^2\frac{\partial}{\partial t}}\,, \end{aligned} \tag{2.2}$$

obeying $D_1^2 = D_2^2 = 0$, $\{D_1, D_2\} = -2i\frac{\partial}{\partial t}$. The theory can be formulated in terms of real superfields, $\Phi$, with the following expansion:

$$\Phi(t, \theta^1, \theta^2) = x(t) + \theta^1\psi(t) + \theta^2\chi(t) + \theta^1\theta^2 F(t)\,, \tag{2.3}$$

containing one auxiliary field $F$ and the physical fields $x$, $\psi$, $\chi$. The continuum action is a full superspace integral:

$$S = \int dt\, d\theta^2 d\theta^1 \left(\frac{1}{2}D_1\Phi D_2\Phi - h(\Phi)\right)$$

$$= \int dt \left( \frac{1}{2}\dot{x}^2 + \frac{1}{2}F^2 - i\psi\,\dot{\chi} - h'(x)F + h''(x)\,\psi\,\chi \right), \tag{2.4}$$

where primes denote derivatives of $h(x)$ with respect to $x$. After eliminating $F$, the component action reads:

$$S = \int dt \left( \frac{1}{2}\dot{x}^2 - \frac{1}{2}h'(x)^2 - i\chi\,\dot{\psi} - \chi\,h''(x)\,\psi \right) \tag{2.5}$$

Eq. (2.4) contains relevant and marginal terms only: $t$ has the usual mass dimension $-1$, and $x(t)$ and $\theta^1, \theta^2$ are assigned mass dimensions $-1/2$. The measure $dt\,d\theta^1\,d\theta^2$ is thus dimensionless as is the lagrangian density; the couplings in the superpotential $h(\Phi)$ are all relevant because the superfield $\Phi$ has negative mass dimension.

As explained in the introduction, naive discretization of the path integral breaks supersymmetry. Our goal now is to discretize the path integral while preserving one of the two nilpotent supercharges, for example $Q_1$. It follows from (2.1) that, upon conjugation, $Q_1$ can be represented as a pure derivative with respect to $\theta^1$:

$$Q \equiv \frac{\partial}{\partial\theta^1} = e^{i\theta^1\theta^2\frac{\partial}{\partial t}}\,Q_1\,e^{-i\theta^1\theta^2\frac{\partial}{\partial t}}. \tag{2.6}$$

Similarly, the superfield $\Phi$ in the conjugated basis is given by:[6]

$$\begin{aligned} \Phi'(t,\theta^1,\theta^2) &= e^{i\theta^1\theta^2\frac{\partial}{\partial t}}\,\Phi(t,\theta^1,\theta^2)\,e^{-i\theta^1\theta^2\frac{\partial}{\partial t}} = \Phi(t + i\theta^1\theta^2, \theta^1, \theta^2) &(2.7)\\ &= (x + \theta^1\psi) + \theta^2(\chi - \theta^1(i\dot{x} + F)). &(2.8) \end{aligned}$$

The action of $Q$ on the field $\Phi'$ is simply a shift of $\theta^1$. As (2.7) shows, the field $\Phi'$ splits into two components, irreducible with respect to the action of $Q$.

Since our lattice action will only preserve the $Q$ supersymmetry, from now on we will denote $\theta^1$ simply by $\theta$. Furthermore, we will also continue $t$ to euclidean space $t \to it$ (to not clutter notation, from now on we will use $t$ to denote euclidean time). In the reduced "superspace" $(t,\theta)$, the components of $\Phi'(t)$ of (2.7), which are irreducible under the $Q$ action, are introduced as new superfields and are denoted by $U$ and $\xi$:

$$U(t) = x(t) + \theta\psi(t), \qquad \xi(t) = \chi(t) - \theta\,(F(t) + \dot{x}(t)). \tag{2.9}$$

The supersymmetry $Q$ now acts as a purely "internal" transformation and no change in its action occurs when we discretize time. Thus, we replace the continuum variable $t$ by a set of discrete discrete points $t^i$, $i = 1, \ldots N$, such that $t^i - t^{i-1} = a$ is the lattice spacing; the total size of the euclidean time circle is thus $Na$ ($t^{i+N} \equiv t^i$). We denote $x(t^i)$ simply by $x^i$, and similarly for the other component fields. We then introduce the discrete version of the superfields (2.9), $U^i$ and $\xi^i$, at every lattice point:

$$U^i = x^i + \theta\psi^i, \qquad \xi^i = \chi^i - \theta\left(F^i + \frac{x^i - x^{i-1}}{a}\right). \tag{2.10}$$

[6]To the reader familiar with 4d $N = 1$ supersymmetry, this conjugation is similar to "going to the chiral basis," where $\bar{D}_{\dot\alpha}$ and $Q_\alpha$ are represented by pure $\theta$-derivatives.

In (2.10), we chose to discretize $\dot{x}$ using a backward lattice derivative. We now see that our superfield $\xi^i$ is slightly nonlocal on the lattice (this will be of recurring interest in our higher-dimensional examples). In 1d, however, we can simply change the bosonic lattice variables $(x^i, F^i)$ to new ones, $(x^i, f^i)$, defined as follows:

$$x^i \to x^i\,, \qquad F^i \to f^i - \frac{x^i - x^{i-1}}{a}\,, \qquad i = 1, \ldots, N\,. \tag{2.11}$$

It is easily seen that the jacobian of this transformation is unity. We then work in terms of the local lattice superfields:

$$U^i = x^i + \theta\psi^i\,, \qquad \xi^i = \chi^i - \theta f^i\,. \tag{2.12}$$

The supersymmetry transformations of the lattice component fields $x^i, \psi^i, \chi^i, f^i$, generated by $Q$ are easily read off eq. (2.12), since $\delta_1 \equiv \epsilon^1 Q = \epsilon^1 \partial/\partial\theta$. The supersymmetry transformation $\delta_2 = \epsilon^2 Q_2$, generated by the second supercharge $Q_2 \equiv \partial/\partial\theta^2 + 2i\theta\partial_t$ in the conjugate basis (2.7), can also be easily worked out. The result for $\delta_1$ and $\delta_2$ is:

$$\begin{aligned} &\delta_1 x^i = \epsilon^1 \psi^i\,, \quad \delta_1 \psi^i = 0\,, && \delta_1 \chi^i = -\epsilon^1 f^i\,, \quad \delta_1 f^i = 0\,, \\ &\delta_2 x^i = \epsilon^2 \chi^i\,, \quad \delta_2 \psi^i = \epsilon^2 \left( f^i - \frac{2}{a}\,\hat{\Delta} x^i \right)\,, && \delta_2 \chi^i = 0\,, \quad \delta_2 f^i = \frac{2}{a}\,\epsilon^2 \hat{\Delta} \chi^i\,, \end{aligned} \tag{2.13}$$

where $\hat{\Delta}$ is a finite difference operator. It can be taken, for example, as the backward difference $\Delta^- x^i = x^i - x^{i-1}$ or the symmetric difference $\Delta^S x^i = (x^{i+1} - x^{i-1})/2$ operator; eq. (2.14) below holds for any linear finite difference. It is straightforward to see that the transformations $\delta_1$ and $\delta_2$ on the lattice fields $x^i$, $\psi^i$, $\chi^i$, and $f^i$, obey a discretized form of the continuum supersymmetry algebra:[7]

$$[\delta_1, \delta_2] = \frac{2}{a}\,\epsilon^1 \epsilon^2\,\hat{\Delta}\,, \qquad \delta_1^2 = \delta_2^2 = 0\,. \tag{2.14}$$

As we argued in the introduction, and will see more explicitly below, only one of the supersymmetries, $\delta_1$, can be exactly preserved by the interactions on the lattice. The variation of the interaction terms under the second supersymmetry $\delta_2$ is of order the lattice spacing, due to the failure of the Leibnitz rule for finite difference operators.

Now we are ready to write the supersymmetric lattice action. We require that the action be bosonic, invariant under the $Q$ supersymmetry, local, and discrete time-translation invariant. Supersymmetry will be respected if our action is an integral over $\theta$ of a function of the superfields $U^i, \xi^i$ and their $\theta$-derivatives. The mass dimensions of $U^i$ and $\xi^i$ are $-1/2$ and 0, respectively.

[7] In higher dimensional examples, e.g. 4d $\mathcal{N} = 1$ supersymmetry and its dimensional reductions, it is similarly straightforward to latticize the algebra (only when acting on linear functions of the fields, as in (2.13), (2.14)). The algebra then closes on lattice translations, provided one uses symmetric differences to replace continuum derivatives (to ensure vanishing of various unwanted $\gamma$ matrix combinations in the commutator).



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Consider first the bilinear candidates for a local superspace action density. The continuum measure, $\int dt d\theta$, is replaced by the lattice sum $a\sum_i \int d\theta$, which is fermionic and has mass dimension $-1/2$. We are thus interested in forming bilinear fermionic terms of mass dimension $1/2$. Denoting by $\Delta$ the backward difference operator, e.g. $\Delta U^i \equiv U^i - U^{i-1}$, we find the following complete list of local bilinear marginal and relevant terms:

1. $\xi^i \frac{\partial}{\partial\theta}\xi^i$ of mass dimension $1/2$ (marginal),
2. $\xi^i \frac{1}{a}\Delta U^i$ of mass dimension $1/2$ (marginal),
3. $\xi^i U^i$ of mass dimension $-1/2$ (relevant),
4. $U^i \frac{\partial}{\partial\theta}U^i$ of mass dimension $-1/2$ (relevant),
5. $\frac{1}{a}\Delta U^i \frac{\partial}{\partial\theta}U^i$ of mass dimension $1/2$ (marginal).

For reasons that will become clear shortly, we will impose an additional global discrete symmetry:

$$\theta \to i\theta\,, \qquad U^i \to U^i\,, \qquad \xi^i \to i\xi^i\,, \tag{2.15}$$

under which the superspace measure transforms as $d\theta \to -id\theta$. The path integral measure $\Pi_i d\psi^i d\chi^i dx^i df^i$ is invariant, as the fermions $\psi$ and $\chi$ transform by $-i$ and $i$, respectively, while the bosons are invariant. The first three terms above get multiplied by $i$ after a symmetry transformation, as required by invariance of the action. The fourth and fifth term get multiplied by $-i$ and the corresponding terms in the action are not invariant. Thus, the most general bilinear local action consistent with the imposed symmetries is given by:

$$S = -\sum_i a \int d\theta \left( \frac{1}{2}\xi^i \frac{\partial}{\partial\theta}\xi^i + \xi^i \frac{1}{a}\Delta U^i + m\,\xi^i\,U^i \right), \tag{2.16}$$

where we have introduced a mass parameter $m$ consistent with the dimension of the relevant term (it is easily seen that we have obtained a latticized form of the action of the supersymmetric oscillator); furthermore, note that rescaling of the superfields and $m$ always permits one to bring the action to the form (2.16).

Let us now study the variation of the quadratic action, eq. (2.16), under the second supersymmetry $\delta_2$ (2.13). It is easy to see that the free action is invariant under both $\delta_1$ and $\delta_2$, provided $\hat{\Delta}$ in (2.13) is taken to be the symmetric difference operator. Note that in the action, we use instead the backward derivative in order to avoid the fermion and boson doubling which appears whenever a symmetric derivative is used. Using the backward finite difference corresponds to adding a supersymmetric Wilson term with a particular value of $r$, as follows from $\Delta^- = \Delta^S - \frac{1}{2}\Delta^2$ (where $\Delta^2 x^i = x^{i+1} + x^{i-1} - 2x^i$ is the lattice laplacian from the Wilson term). Any value of $r$ is consistent with both $\delta_1$ and $\delta_2$ invariances.

Introducing a superpotential interaction $h(x)$ in (2.16) is now trivial: one simply replaces $mU^i \to h'(U^i)$ in the last term:

$$S = -\sum_i a \int d\theta \left( \frac{1}{2} \xi^i \frac{\partial}{\partial\theta} \xi^i + \xi^i \frac{1}{a} \Delta U^i + \xi^i h'(U^i) \right). \tag{2.17}$$

After eliminating the auxiliary field $f^i$ via its equation of motion, the component action becomes:

$$S = \sum_{i=1}^{N} a \left( \frac{1}{2} \left( h'(x^i) + \frac{x^i - x^{i-1}}{a} \right)^2 + \chi^i h''(x^i) \psi^i + \chi^i \frac{\psi^i - \psi^{i-1}}{a} \right). \tag{2.18}$$

This is nothing other than the action introduced in [13], denoted there as "$S_{new}$." The action was also studied in [14], and related to a topological field theory construction in [16].

The action (2.18) can now be used to define the following supersymmetric lattice partition function:

$$Z_N = -c^N \prod_{i=1}^{N} \int d\chi^i d\psi^i \int_{-\infty}^{\infty} dx^i \, e^{-S} , \tag{2.19}$$

where $c^N$ a normalization constant, $c = (2\pi a)^{-1/2}$, and a minus sign is included for convenience (see appendix A). The supersymmetric lattice partition function $Z_N$ has a number of nice properties, discussed in detail in appendix A. We enumerate the results below, along with some comments:

1. Despite the fact that the action (2.18) is not reflection positive, in the small-$a$ limit $Z_N$ defines a hermitian hamiltonian, $H_{SQM} = (\hat{p}^2 + h'(\hat{q})^2 - h''(\hat{q})[\hat{b}^\dagger, \hat{b}])/2$, acting in a positive norm Hilbert space. We show this using the transfer matrix formalism, see eqs. (A.3)–(A.7). Thus, in the small lattice spacing limit, $Z_N$ approaches the Witten index of the supersymmetric quantum mechanics:

$$\lim_{N\to\infty, a\to 0} Z_N = \mathrm{tr}(-1)^F e^{-\beta H_{SQM}} , \qquad \beta \equiv Na - \text{ fixed} . \tag{2.20}$$

2. Moreover, we also show that the exact supersymmetry of the partition function (2.19) can be used to exactly compute $Z_N$ for any superpotential and for any value of $N$, $a$. We show that $Z_N$ coincides with the continuum theory Witten index already at finite lattice spacing, see eqs. (A.8)–(A.9).

3. The transfer matrix representation can also be used to give a lattice formulation of the finite-temperature partition function $Z_N(\beta) = \mathrm{tr}\, e^{-\beta H_{SQM}}$. In terms of the lattice action and functional measure, this amounts to antiperiodic boundary conditions for the fermions. The exact supersymmetry of $Z_N(\beta)$ is now broken globally by the boundary conditions but is locally present and recovered in the low temperature limit. (In contrast to $Z_N$, $Z_N(\beta)$ is generally not exactly calculable.)



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4. Another consequence of the exact supersymmetry of the partition function is the following. Numerical simulations of correlation functions using the supersymmetric partition function $Z_N$ have shown [13] convergence to the fully supersymmetric continuum limit; by contrast, similar simulations using a naive nonsupersymmetric discretization do not. In the case of the supersymmetric $Z_N$, the spectrum is degenerate at any value of $a$ and $N$ and some Ward identities associated with the nonexact supersymmetry have been observed to hold within numerical error [13, 14].

We pause to note that had we added the $U^{i-1}\frac{\partial}{\partial\theta}U^i$ term (the only nonvanishing bilinear term forbidden by the extra discrete symmetry (2.15)), which gives rise to an extra $\psi^i\psi^{i-1}$ term in the component action (2.18), we would have also obtained a supersymmetric discretized partition function. One can show, however, using the transfer matrix formalism, that its continuum limit does not define a hermitean hamiltonian system.

One final comment, which has already been emphasized in [13, 14, 16], is that if we introduce the new "Nicolai variable:"

$$\mathcal{N}^i(x) \equiv h'(x^i) + \frac{x^i - x^{i-1}}{a}\,, \tag{2.21}$$

we can rewrite (2.18) in the following form:

$$S = \sum_{i=1}^{N} a \left( \frac{1}{2}\mathcal{N}^i(x)^2 + \chi^i \, \frac{\partial \mathcal{N}^i(x)}{\partial x^j}\, \psi^j \right), \tag{2.22}$$

where a sum over repeated indices in the fermion term is understood. This form of the supersymmetric lattice action for supersymmetric quantum mechanics has been obtained before by discretization of the continuum Nicolai variables. We note, however, that our derivation of the supersymmetric lattice action assumed no prior knowledge of the existence of a local Nicolai map. Thus, we expect that our "lattice superfield" approach is more general and can be applied also to systems where a local Nicolai map is not known, for example quantum mechanics on non-flat manifolds; we leave this exercise for future study.

We now continue with the application of our formalism to higher dimensional models.

## 3. $(1,1)$ 2d and $\mathcal{N}=1$ 3d supersymmetry

The next step is to consider higher dimensional theories also with two supercharges. Such Lorentz invariant theories exist in two and three dimensions; $(1,0)$ theories also exist in 2d but have no nilpotent supercharge. In this section, we review arguments (for completeness) demonstrating that in $(1,1)$ 2d and minimal ($\mathcal{N}=1$) 3d supersymmetry a nilpotent supercharge does not exist. Thus, $Q$-supersymmetry as a path to exact lattice supersymmetry is not an option in these cases. Nevertheless, the number of fine-tunings required in a nonsupersymmetric lattice theory may be small, or not needed at all, due to the super-renormalizability of some of these theories, see [18]. We will not consider these theories further, as our focus here is on supersymmetric lattice actions.

The $(1,1)$ 2d algebra can be written ($\alpha, \beta, \mu = 1, 2$) as:

$$\{Q_\alpha, Q_\beta\} = 2i\gamma^\mu_{\alpha\beta}\partial_\mu\,. \tag{3.1}$$

Symmetry of the l.h.s. requires that $\gamma^\mu$ be symmetric. Without loss of generality we may choose $\gamma^1 = \sigma_1$. It follows from the symmetry requirement and the euclidean Clifford algebra that (up to a sign) $\gamma_2 = \sigma_3$. Neither supercharge is nilpotent. Suppose we form the most general linear combination:

$$Q = c_1 Q_1 + c_2 Q_2 \,. \tag{3.2}$$

Then it follows that:

$$Q^2 = 2ic_1c_2\partial_1 + i(c_1^2 - c_2^2)\partial_2 \,. \tag{3.3}$$

Clearly $Q^2 = 0$ has only the trivial solution $c_1 = c_2 = 0$. The case of lorentzian metric is simply obtained by the identifications $\gamma_2 = i\gamma_0$ and $\partial_2 = -i\partial_0$. The impossibility of nontrivial nilpotent supercharge is unchanged.

In 3d the $\mathcal{N} = 1$ algebra can be written ($\alpha, \beta = 1, 2$ and $\mu = 1, 2, 3$):

$$\{Q_\alpha, Q_\beta\} = 2i(\epsilon\gamma^\mu)_{\alpha\beta}\partial_\mu \,, \tag{3.4}$$

where $\epsilon = i\sigma_2$. Symmetry of the l.h.s. requires that $\epsilon\gamma^\mu$ be symmetric. Without loss of generality this condition and the euclidean Clifford algebra for $\gamma^\mu$ can be satisfied by choosing $\gamma^\mu = \sigma_\mu$. Once again neither of the supercharges are nilpotent. Furthermore we find that:

$$Q^2 = i(c_1^2 - c_2^2)\partial_1 - (c_1^2 + c_2^2)\partial_2 - 2ic_1c_2\partial_3 \,. \tag{3.5}$$

Again, $Q^2 = 0$ has only the trivial solution $c_1 = c_2 = 0$. For a lorentzian metric, one simply takes $\gamma_3 = i\gamma_0$ and $\partial_3 = -i\partial_0$, to arrive at the conclusion that there is no nilpotent supercharge in this case as well.

## 4. $(2,0)$ 2d theory and related 3d and 4d theories

Continuing our survey of supersymmetric theories, we note that another theory with only two supercharges exists only in 2d — the $(2,0)$ theory. It provides an important example of a theory where nilpotent supercharges exist, i.e. a theory that obeys criterion 1.) of the introduction, but violates criterion 2.). Hence, not all continuum interactions preserve the nilpotent supersymmetry on the lattice. A similar conclusion also holds for the 4d $\mathcal{N} = 1$ Wess-Zumino model and for its compactification to 3d.

The $(2,0)$ algebra (here and in the following sections, our notation for 2d supersymmetry is as in, e.g. [23]) is generated by two supercharges $Q_+$ and $\bar{Q}_+$, obeying:

$$\{Q_+, \bar{Q}_+\} = -2i\partial_+ \,, \qquad Q_+^2 = \bar{Q}_+^2 = 0 \,, \qquad \partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1) \,. \tag{4.1}$$

The supercharges and covariant derivatives can be represented by differential operators acting on the $(x^\pm, \theta^+, \bar{\theta}^+)$ superspace as:

$$\begin{aligned} Q_+ &= \frac{\partial}{\partial\theta^+} + i\bar{\theta}^+\partial_+ \,, \qquad & \bar{Q}_+ &= -\frac{\partial}{\partial\bar{\theta}^+} - i\theta^+\partial_+ \\ D_+ &= \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+\partial_+ \,, \qquad & \bar{D}_+ &= -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+\partial_+ \,. \end{aligned} \tag{4.2}$$

We will consider, as in all cases in this paper, theories of scalar and fermion fields. In the continuum $(2,0)$ theories, these fall into chiral scalar, $\Phi$, and chiral fermion, $\Psi_-$, multiplets. Chirality means, as usual, that they are subject to the supersymmetric constraint: $\bar{D}_+\Phi = 0$, $\bar{D}_+\Psi_- = 0$ (the complex conjugates of $\Phi$ and $\Psi_-$ are antichiral). The chiral scalar and chiral fermion multiplets have the following component expansions (and similar for their antichiral complex conjugates):

$$\begin{aligned} \Phi &= \phi + \theta^+\psi_+ - i\theta^+\bar{\theta}^+\partial_+\phi \\ \Psi_- &= \psi_- + \theta^+ G - i\theta^+\bar{\theta}^+\partial_+\psi_- \,, \end{aligned} \tag{4.3}$$

where $\phi$ is a complex scalar field, $\psi_\pm$ — physical fermion fields and $G$ — an auxiliary field. The continuum invariant actions are given in terms of "$D$" and "$F$" terms:

$$\begin{aligned} L_D &= \int d^2x \int d\theta^+ d\bar{\theta}^+ \left( i\bar{\Phi}\partial_-\Phi + \bar{\Psi}_-\Psi_- \right) \\ L_F &= \int d^2x \int d\theta^+ \Psi_- V(\Phi)\Big|_{\bar{\theta}^+=0} \\ L_{\bar{F}} &= \int d^2x \int d\bar{\theta}^+ \bar{\Psi}_- V(\bar{\Phi})\Big|_{\theta^+=0} \,. \end{aligned} \tag{4.4}$$

Our latticization of (4.4) will proceed in complete analogy with the quantum mechanical example and so we omit many of the tedious steps.

We choose to preserve one of the nilpotent generators on the lattice, say $Q_+$. Just as in the quantum mechanics case, we transform to a basis where $Q_+$ is given by $\partial/\partial\theta^+$ and to euclidean signature, by defining $x^0 = -ix^2$, $z = x^1 + ix^2$, $\partial_+ = \partial_{\bar{z}}$, $\partial_- = -\partial_z$. We also replace the continuum by a 2d square lattice and the derivatives by appropriate finite differences. Thus, we find that the superfields $\Phi, \bar{\Phi}, \Psi_-$, and $\bar{\Psi}_-$ decompose into the following "lattice superfield" components, irreducible under the $Q_+$ action:

$$\begin{aligned} \Phi &\to \phi + \theta^+\psi_+ \,, \\ \bar{\Phi} &\to \bar{\phi} \,, \\ &\quad\ \bar{\psi}_+ - 2i\theta^+\Delta_{\bar{z}}\bar{\phi} \\ \Psi_- &\to \psi_- + \theta^+ G \,, \\ \bar{\Psi}_- &\to \bar{\psi}_- \,, \\ &\quad\ \bar{G} - 2i\theta^+\Delta_{\bar{z}}\bar{\psi}_- \,, \end{aligned} \tag{4.5}$$

where $\Delta_{\bar{z}}$ is a discretization of $\partial_{\bar{z}}$ (every line on the r.h.s. represents a superfield irreducible under the $Q_+$ action). The supersymmetry transformations of the component fields are immediately read off eq. (4.5), as the $Q_+$ transformations are simple shifts in $\theta^+$; thus, for example, $\bar{\phi}$ is $Q_+$ invariant, while $\delta_{\epsilon_+}\bar{\psi}_+ = -2i\epsilon_+\Delta_z\bar{\phi}$, etc.

Using these superfields, it is straightforward to see that the $L_D$ and $L_F$ terms can be written on the lattice in a form that preserves the $Q_+$ supersymmetry, i.e. shifts in $\theta^+$; we do not give the details since this is not our main point here.

More important is the fact that the $L_{\bar{F}}$ interaction on the lattice can not be written in a $Q_+$ invariant form. This is because $L_{\bar{F}}$ is written as an integral over the antichiral $(2,0)$

superspace and its $Q_+$ variation requires use of the Leibnitz rule. To see this, notice that from (4.4), we find, in the continuum:

$$L_{\bar{F}} = \int d^2x \left(\bar{G}V(\bar{\phi}) + \bar{\psi}_-\bar{\psi}_+V'(\bar{\phi})\right) . \tag{4.6}$$

Under $Q_+$ transformations $\bar{\phi}$ and $\bar{\psi}_-$ are inert, see (4.5); however, $\bar{\psi}_+$ and $\bar{G}$ shift into derivatives of $\bar{\phi}$ and $\bar{\psi}_-$, respectively. Thus, the $Q_+$ variation of $L_{\bar{F}}$ is proportional to $\partial_{\bar{z}}\bar{\psi}_-V(\bar{\phi}) + \bar{\psi}_-V'(\bar{\phi})\partial_{\bar{z}}\bar{\phi} = \partial_{\bar{z}}\left(\bar{\psi}_-V(\bar{\phi})\right)$, and hence is a total derivative. Thus, the invariance of the antichiral superspace integral, even under the nilpotent $Q_+$, requires the validity of the Leibnitz rule. Hence we can not latticize the action (4.4) in a manner that preserves supersymmetry of the antichiral interaction terms.[8]

The situation with the $\bar{F}$ terms the $(2,0)$ theory is repeated in a number of other two-, three-, and four-dimensional models, where nilpotent supercharges exist — the "B"-choice of lattice supercharges in $(2,2)$ 2d theories (see following section and appendix B), the $\mathcal{N}=2$ 3d theory, and the $\mathcal{N}=1$ 4d Wess-Zumino model. In all these four-supercharge cases, in standard notation, two anticommuting nilpotent supercharges exist and can be chosen to be, say, the $Q_\alpha$ ($\alpha = 1,2$ is the $\mathrm{SL}(2,\mathbb{C})$ index). Similar to our discussion above, the antichiral part of the interaction lagrangian in these models can not be latticized in a manner preserving the nilpotent supercharges, since vanishing of its $Q_\alpha$ variation requires validity of the Leibnitz rule.

## 5. $(2,2)$ 2d supersymmetry: the $A$-type supersymmetric lattice

### 5.1 Nilpotent charges, superfields, and global symmetries

We are thus led to consider theories with four supercharges in two dimensions, which will be the main focus of this paper. The $(2,2)$ 2d supersymmetry algebra with no central charges,[9] in Minkowski space, is simply the dimensional reduction of the $\mathcal{N}=1$ supersymmetry algebra in four dimensions:

$$\{Q_\pm, \bar{Q}_\pm\} = -2i\partial_\pm , \quad Q_\pm^2 = \bar{Q}_\pm^2 = 0 , \quad Q_\pm^\dagger = \bar{Q}_\pm , \quad \{Q_+, Q_-\} = 0 , \qquad \partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1) . \tag{5.1}$$

As explained in the introduction, if we were to construct a lattice action of an interacting supersymmetric theory, the best we can hope for is to explicitly preserve only a subset of the four supercharges — those, whose anticommutators do not involve translations. Clearly, from eq. (5.1), in the $(2,2)$ case, the maximal number of nilpotent anticommuting supersymmetry generators is two.

There are two, up to hermitean conjugation, possible choices of nilpotent anticommuting generators. We will call them "$A$-type," taking the $\bar{Q}_+$, $Q_-$ pair of generators, and "$B$-type," taking the $\bar{Q}_+, \bar{Q}_-$ pair of supercharges. The $B$-type choice is also possible

[8] One "loophole," which we leave for future work is the construction of $Q_+$-invariant $(2,0)$ sigma model lattice actions (they only have $D$-type interactions, so the "$\bar{F}$-term obstruction" does not apply) and the study of related issues of doublers and anomalies, which are likely to arise, as in $(2,2)$ sigma models.

[9] For a lattice realization of a theory with nonzero central charges, see appendix C.

in the 3d $\mathcal{N}=2$ and 4d $\mathcal{N}=1$ theories; the difficulties that this choice of exact lattice supersymmetries faces were already discussed in the previous section.

Thus, from now on we focus on the $A$ choice. This choice is unique to 2d, since $Q_-$ and $\bar{Q}_+$ cease to anticommute when the algebra (5.1) is uplifted to 3d and 4d — their anticommutator involves a translation in the "extra" spatial directions. Thus, in 3d and 4d they can not be simultaneously brought into a form involving only $\theta$ derivatives.

We will study theories, whose continuum limit are theories of chiral superfields only: $(2,2)$ sigma models or Landau-Ginzburg (LG) models. To address the task of writing the most general action consistent with the exact lattice supersymmetry, we will develop a "lattice superfield" formalism, making the study of invariant interactions and their symmetries extremely straightforward. We follow the same steps as in the quantum mechanics case. Since, conceptually, there are no new steps involved, but the notation is significantly messier, we give all details in appendix B.

Just as in the quantum mechanics case of section 2, the end result of the "lattice superfield" construction is the introduction of the following $A$-type lattice superfields, which correspond to the irreducible (under the $Q_-$ and $\bar{Q}_+$ action) components of continuum chiral superfields:

$$\begin{aligned} U_{\vec{m}} &= \phi_{\vec{m}} + \theta^- \psi_{-,\vec{m}}\,, \\ \Xi_{\vec{m}} &= \psi_{+,\vec{m}} - \bar{\theta}^+ i\Delta_{\bar{z}}\phi_{\vec{m}}\theta^- F_{\vec{m}} + i\theta^-\bar{\theta}^+\Delta_{\bar{z}}\psi_{-,\vec{m}}\,, \\ \bar{U}_{\vec{m}} &= \bar{\phi}_{\vec{m}} - \bar{\theta}^+\bar{\psi}_{+,\vec{m}}\,, \\ \bar{\Xi}_{\vec{m}} &= \bar{\psi}_{-,\vec{m}} - \theta^- i\Delta_z\bar{\phi}_{\vec{m}} - \bar{\theta}^+\bar{F}_{\vec{m}} + i\theta^-\bar{\theta}^+\Delta_z\bar{\psi}_{+,\vec{m}}\,. \end{aligned} \tag{5.2}$$

We note that the fermionic lattice superfields $\Xi$ and $\bar{\Xi}$ are slightly nonlocal on the lattice, because of the appearance of finite difference operators, and that the fields $U, \Xi$ are not independent, as $\Xi = \psi_+ + \theta^- F - \bar{\theta}^+ i\Delta_{\bar{z}}U$. This is because $U$ and $\Xi$ are the components of a continuum chiral superfield in the appropriate basis. As explained in appendix B, we denote $\Delta_z \equiv \Delta_1 - i\Delta_2$, and $\Delta_{\bar{z}} \equiv \Delta_1 + i\Delta_2$, where $\Delta_i$ is a finite difference operator in the $i$-th direction. As in quantum mechanics, the supersymmetry transformations of the lattice superfields under $Q_-$, $\bar{Q}_+$ are simply read off eq. (5.2), since the corresponding supersymmetries are now simply shifts of $\theta^-$ and $\bar{\theta}^+$.

Any function of the lattice superfields (5.2) integrated over the $\theta^-$, $\bar{\theta}^+$ superspace is then trivially invariant under the two supersymmetries. However, as we will see shortly (in section 2, we already saw an example in the simpler case of quantum mechanics), many of the superspace lattice invariants do not correspond to continuum $(2,2)$ supersymmetric and euclidean rotation invariant actions. This should not come as a surprise, as the preservation of only the $Q_-$ and $\bar{Q}_+$ supersymmetries is not restrictive enough to recover all continuum limit symmetries. In order to achieve the desired $(2,2)$-supersymmetric continuum limit, we have to impose additional symmetries on the lattice actions.

To this end, recall that $(2,2)$ supersymmetric continuum theories have three classical global U(1) symmetries: euclidean rotation, $\mathrm{U}(1)_E$, as well as the vector and axial $\mathrm{U}(1)_{V/A}$ transformations, whose action is:

$$\mathrm{U}(1)_V:\ \ \psi_\pm \to e^{i\beta}\psi_\pm\,, \quad \bar{\psi}_\pm \to e^{-i\beta}\bar{\psi}_\pm\,, \quad F \to e^{2i\beta}F\,, \quad \bar{F} \to e^{-2i\beta}\bar{F}$$

$$\mathrm{U}(1)_A: \ \psi_\pm \to e^{\pm i\omega}\psi_\pm\,, \quad \bar\psi_\pm \to e^{\mp i\omega}\bar\psi_\pm, \tag{5.3}$$

and trivial on other fields. The vector and axial transformations above are obtained by assigning zero $V$ and $A$ charge to the continuum superfields, as is indicated by the trivial action on the scalar components in (5.3). The continuum $\mathrm{U}(1)_E$ is, of course, broken on the lattice. On a square lattice, a discrete $Z_4$ subgroup can be preserved, with action, denoting by $\Phi$ any of the bosonic fields, given by:

$$\begin{aligned} Z_4: \ & \Phi_{m_1,m_2} \to \Phi_{m_2,-m_1}\,, \\ & \psi_{\pm,m_1,m_2} \to e^{\mp\frac{i\pi}{4}}\psi_{\pm,m_2,-m_1}\,, \\ & \bar\psi_{\pm,m_1,m_2} \to e^{\mp\frac{i\pi}{4}}\,\bar\psi_{\pm,m_2,-m_1}\,. \end{aligned} \tag{5.4}$$

Two additional symmetries will be important for us. Fermion parity is defined in the usual manner:

$$Z_{2F}: \ \theta^- \to -\theta^-\,, \quad \bar\theta^+ \to -\bar\theta^+\,, \quad \Xi \to -\Xi\,, \quad \bar\Xi \to -\bar\Xi\,, \quad U \to U\,, \quad \bar U \to \bar U\,. \tag{5.5}$$

Finally, we define an involution (hereafter called $I$) transformation on the lattice action, which includes complex conjugation of the parameters in the action and c-numbers in superfields, a reversal of the order of fermionic fields, and subsequent replacements of all lattice component fields as follows:

$$\begin{aligned} I: \ & \theta^- \to -i\bar\theta^+\,, \quad \bar\theta^+ \to -i\theta^-\,, \quad \psi_\pm \to i\bar\psi_\mp\,, \quad \bar\psi_\pm \to i\psi_\mp\,, \\ & \phi \to \bar\phi\,, \quad \bar\phi \to \phi\,, \quad F \to \bar F\,, \quad \bar F \to F\,. \end{aligned} \tag{5.6}$$

The action of these symmetries on the lattice superfields (5.2) is summarized below for further use:

$$\begin{aligned} \mathrm{U}(1)_V: \ & \theta^- \to e^{-i\beta}\,\theta^-\,, \quad \bar\theta^+ \to e^{i\beta}\,\bar\theta^+\,, \quad \Xi \to e^{i\beta}\,\Xi\,, \quad \bar\Xi \to e^{-i\beta}\,\bar\Xi\,, \\ & U \to U\,, \quad \bar U \to \bar U\,, \\ \mathrm{U}(1)_A: \ & \theta^- \to e^{i\omega}\,\theta^-\,, \quad \bar\theta^+ \to e^{i\omega}\,\bar\theta^+\,, \quad \Xi \to e^{i\omega}\,\Xi\,, \quad \bar\Xi \to e^{i\omega}\,\Xi\,, \\ & U \to U\,, \quad \bar U \to \bar U\,, \\ Z_4: \ & \theta^- \to e^{-i\frac{\pi}{4}}\theta^-\,, \quad \bar\theta^+ \to e^{i\frac{\pi}{4}}\bar\theta^+\,, \quad \Xi_{m_1,m_2} \to e^{-i\frac{\pi}{4}}\Xi_{m_2,-m_1}\,, \\ & \bar\Xi_{m_1,m_2} \to e^{i\frac{\pi}{4}}\bar\Xi_{m_2,-m_1}\,, \quad U_{m_1,m_2} \to U_{m_2,-m_1}\,, \\ & \bar U_{m_1,m_2} \to \bar U_{m_2,-m_1}\,, \\ Z_{2F}: \ & \theta^- \to -\theta^-\,, \quad \bar\theta^+ \to -\bar\theta^+\,, \quad \Xi \to -\Xi\,, \quad \bar\Xi \to -\bar\Xi\,, \\ & U \to U\,, \quad \bar U \to \bar U\,, \\ I: \Xi \to i\,\bar\Xi\,, \quad & \bar\Xi \to i\,\Xi\,, \quad U \to \bar U\,, \quad \bar U \to U\,, \quad d\theta^- d\bar\theta^+ \to -d\theta^- d\bar\theta^+\,. \end{aligned} \tag{5.7}$$

In the next section, we will use these symmetries to restrict the possible terms that can appear in the action. We stress again that the transformation properties of the lattice superfields under the involution $I$ hold for any definition of the lattice derivatives, because the involution interchanges $\psi_+$ and $\bar\psi_-$ (and $\psi_-$ with $\bar\psi_+$). In this paper, we will take them to be the symmetric difference operators.

### 5.2 The $(2,2)$ chiral sigma model: supersymmetric lattice action

We now turn to the lattice theory of chiral superfields, preserving the $A$-type supercharges, by classifying the terms allowed by the lattice supersymmetry and other imposed global symmetries.

Consider first superspace invariants made only of $U$ and $\bar{U}$. The simplest ones are half-superspace integrals of functions of $U$ or $\bar{U}$. The terms $\int d\theta^- V(U)$ as well as $\int d\bar{\theta}^+ V^*(\bar{U})$ are clearly supersymmetric, but violate fermion parity. The next simplest possibility is to allow for a full superspace integral (and also allow for superspace derivatives of the superfields to appear in the action) and consider lattice supersymmetry invariants of the form:

$$\int d\theta^- d\bar{\theta}^+ f\left(U, \bar{U}, \frac{\partial U}{\partial \theta^-}, \frac{\partial \bar{U}}{\partial \bar{\theta}^+}\right), \tag{5.8}$$

where we do not indicate the dependence on the lattice points (allowing, at this point, for generic local or nonlocal interactions). Interactions like (5.8) can be arranged to preserve the $\mathrm{U}(1)_V$, $Z_{2F}$, and $I$ (if appropriate complex conjugation conditions are imposed on $f$) symmetries; they can also be made $Z_4$ invariant. Local (or nonlocal) terms like (5.8) can thus only be forbidden by imposing the $\mathrm{U}(1)_A$ as no interaction of the above form preserves the axial symmetry.

The $\mathrm{U}(1)_A$ requires that any full superspace invariant be quadratic in $\Xi$ or $\bar{\Xi}$, as these are the only fields that have axial charges opposite that of the superspace measure. Thus, we now allow the possibility of having also $\Xi$ and $\bar{\Xi}$ appear in the lattice action. Consider the most general $\mathrm{U}(1)_V$, $\mathrm{U}(1)_A$, $Z_{2F}$, $I$, and $Z_4$ invariant lattice superfield action. Also, it suffices to consider only invariants that do not involve $\theta$-derivatives of either $U, \bar{U}$ or $\Xi$, $\bar{\Xi}$; by dimensional analysis, such interactions correspond to irrelevant terms in the continuum limit. The most general lattice action with these symmetries, which is a *local* function of the superfields (5.2) and does not involve superspace derivatives of the fields, is:

$$S_D = -a^2 \sum_{\vec{m}} \int d\bar{\theta}^+ d\theta^- K_{I\bar{J}}(U_{\vec{m}}, \bar{U}_{\vec{m}})\, \bar{\Xi}^{\bar{J}}_{\vec{m}}\, \Xi^I_{\vec{m}}\,, \tag{5.9}$$

where we generalized to the multifield case and absorbed $1/a$ factors in the definition of $\Delta_{z,\bar{z}}$ appearing in $\Xi, \bar{\Xi}$.

We note that the axial, fermion number, and involution invariance also allow *nonlocal* supersymmetric terms without $\theta$-derivatives of the following form:

$$\int d\bar{\theta}^+ d\theta^- \left(\Xi_{\vec{m}}\, \Xi_{\vec{n}} - \bar{\Xi}_{\vec{m}} \bar{\Xi}_{\vec{n}}\right) f(U, \bar{U})\,, \tag{5.10}$$

for $f(U, \bar{U})$ is an arbitrary function of $U, \bar{U}$, which is odd under the involution. The term (5.10) vanishes if $\vec{m} = \vec{n}$ due to the fermionic nature of $\Xi$. Nonlocality, combined with dimensional analysis implies that terms of the form (5.10) are irrelevant in the continuum limit.

Thus, we conclude that the most general local action invariant under $\mathrm{U}(1)_V$, $\mathrm{U}(1)_A$, $Z_{2F}$, $I$, and $Z_4$, and not including superspace derivatives, is given by (5.9). The component expression is immediately seen, upon expansion in superspace, to be:

$$\begin{aligned} S_D = -a^2 \sum_{\vec{m}} & -K_{I\bar{J}}\, \Delta_z \phi^I_{\vec{m}} \cdot \Delta_{\bar{z}} \bar{\phi}^{\bar{J}}_{\vec{m}} + K_{I\bar{J}}\, F^I_{\vec{m}} \bar{F}^{\bar{J}}_{\vec{m}} + \\ & + i K_{I\bar{J}}\, \bar{\psi}^{\bar{J}}_{-,\vec{m}} \left[ \Delta_z \psi^I_{-,\vec{m}} + K^{I\bar{Q}} K_{ML\bar{Q}} \Delta_z \phi^M_{\vec{m}} \cdot \psi^L_{-,\vec{m}} \right] - \\ & - i K_{I\bar{J}}\, \psi^I_{+,\vec{m}} \left[ \Delta_{\bar{z}} \bar{\psi}^{\bar{J}}_{+,\vec{m}} + K^{\bar{J}L} K_{L\bar{M}\bar{Q}} \Delta_{\bar{z}} \bar{\phi}^{\bar{M}}_{\vec{m}} \cdot \bar{\psi}^{\bar{Q}}_{+,\vec{m}} \right] + \\ & + K_{I\bar{J}\bar{L}} F^I_{\vec{m}} \bar{\psi}^{\bar{L}}_{+,\vec{m}} \bar{\psi}^{\bar{J}}_{-,\vec{m}} + K_{\bar{I}JL} \bar{F}^{\bar{I}}_{\vec{m}} \psi^J_{-,\vec{m}} \psi^L_{+,\vec{m}} + \\ & + K_{IL\bar{J}\bar{M}} \bar{\psi}^{\bar{M}}_{+,\vec{m}} \psi^L_{-,\vec{m}} \psi^I_{+,\vec{m}} \bar{\psi}^{\bar{J}}_{-,\vec{m}} \,. \end{aligned} \tag{5.11}$$

For simplicity of notation we denote by $K_{I\bar{J}K}$ the derivative of $K_{I\bar{J}}$ with respect to $U^K$, and similar for higher derivatives, while $K^{I\bar{J}}$ is the matrix inverse to $K_{I\bar{J}}$, in the usual continuum notation. The fermion kinetic terms of $\psi_-, \bar{\psi}_-$ can be written in terms of the Kähler connection $\Gamma^I_{JK} = K^{I\bar{M}} K_{J\bar{M},K}$ (and its complex conjugate for $\psi_+, \bar{\psi}_+$). All quantities, $K_{I\bar{J}}$, etc., are functions of the scalar fields $\phi_{\vec{m}}, \bar{\phi}_{\vec{m}}$ as indicated in (5.9).

The naive continuum limit of (5.11) is the $(2,2)$ nonlinear sigma model action, invariant under the $\mathrm{U}(1)_E$, $\mathrm{U}(1)_V$, $\mathrm{U}(1)_A$, and involution transformations. However, at this point, it should be clear that eq. (5.9), and the corresponding finite-dimensional integral over the components of the lattice superfields, with the $Q_-$, $\bar{Q}_+$-invariant measure:

$$\prod_{\vec{m}} \prod_I dF^I d\bar{F}^{\bar{I}} d\phi^I d\bar{\phi}^{\bar{I}} d\psi^I_- d\bar{\psi}^{\bar{I}}_- d\psi^I_+ d\bar{\psi}^{\bar{I}}_+ \,, \tag{5.12}$$

does not define the desired continuum theory. To see this, note that we have constructed a fully regulated and supersymmetric lattice version of the sigma model, where the continuum global symmetries $\mathrm{U}(1)_A$ and $\mathrm{U}(1)_V$ are both manifest on the lattice. On the other hand, non-Ricci-flat continuum sigma models exhibit an anomaly of the axial symmetry, proportional to the first Chern class of the Kähler manifold. A regulated version which preserves the $\mathrm{U}(1)_A$ can not account for the anomaly. There is no puzzle here, of course: a study of the perturbative spectrum of (5.11), see appendix D, reveals the presence of doublers, which lead to cancellation of the anomaly. This is of course a direct consequence of the Nielsen-Ninomiya theorem [24]. To address these issues, in the following sections (see also appendix C and D) we consider in more detail the fermion kinetic terms in eq. (5.11), their symmetry, and the definition of the lattice partition function of the theory. We note that a version of the lattice action (5.11) for nonlinear sigma models was written before, using a different formalism [17].

Let us now summarize the results of this section. Using our lattice superfield formalism, we constructed a lattice action of the chiral superfield 2d $(2,2)$ sigma model, which exactly preserves two of the continuum supercharges. The action (5.11) is the most general local action consistent with the imposed symmetries, to leading order in the derivative expansion, and for any choice of lattice derivatives. Upon inspection, it is easy to see that the naive continuum limit of this action coincides with the continuum $(2,2)$ nonlinear sigma model



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action. An analysis of the perturbative spectrum of (5.11) reveals the presence of fermionic (and bosonic, because of the lattice supersymmetry) doublers in the spectrum. We thus have to study various ways to lift the doublers and discuss issues related to the definition of the partition function and the quantum continuum limit of the theory.[10]

One possibility[11] is to use "twisted" nonlocal mass (i.e., Wilson) terms to lift the doublers. In appendix C, we describe this construction using our formalism. While this deformation allows the doublers to be lifted and also breaks the anomalous $\mathrm{U}(1)_A$, it explicitly violates the exact lattice supersymmetry. Since a generic $(2,2)$ sigma model is a renormalizable theory (rather than a superrenormalizable one) with logarithmic divergences arising at every order in perturbation theory, it is not immediately obvious whether the supersymmetric continuum limit can be achieved without fine tuning within this framework; this question deserves further study.

The second mechanism to lift the doublers uses superpotential nonlocal mass terms. These terms preserve an exact lattice supersymmetry, but explicitly violate some of the global symmetries. Since some amount of exact supersymmetry is likely to be helpful in achieving the supersymmetric continuum limit, we devote the rest of the paper to studying the $F$-term deformations and the approach to the continuum limit.

### 5.3 Superpotential, $F$-type Wilson terms, and the continuum limit

As just mentioned, the way to lift the doublers while preserving some exact supersymmetry on the lattice is to incorporate a Wilson mass term via a superpotential. A $Q_-$ and $\bar{Q}_+$ supersymmetric $F$-term is not among the $\mathrm{U}(1)_A$, $\mathrm{U}(1)_V$, $Z_4$ and $I$ invariants we've already listed. An $F$-term superpotential interaction, including a doubler-lifting nonlocal Wilson mass term, can be written only if we allow the supersymmetry to be broken to a linear combination of $Q_-$ and $\bar{Q}_+$. We note that this is accord with our general criterion 2.) from the introduction and with our discussion of the $(2,0)$ theory of section 4: a superpotential interaction is an integral over (anti) chiral superspace. The chiral integral's variation under the antichiral supercharge is a total derivative and vice versa. This was the reason that in the $(2,0)$ model an invariant superpotential interaction was not possible. Here, however, we have the opportunity to preserve a linear combination of the chiral and antichiral $Q_-$ and $\bar{Q}_+$ (in other words we let the total derivative variation of the superpotential under $\bar{Q}_+$ cancel that of the anti-superpotential under $Q_-$).

Thus, here we generalize our construction to lattice models where the only supersymmetry is the one generated by $Q_A = Q_- + \bar{Q}_+$, i.e. by $\hat{\delta}$ of eq. (B.2) with $\epsilon^- = -\bar{\epsilon}^+$ (and $\epsilon^+ = \bar{\epsilon}^- = 0$). The explicit breaking to the diagonal supersymmetry can be easily incorporated by introducing a spurion superfield $\Xi_0, \bar{\Xi}_0$ with the following $\theta$-dependent "vev:"

$$\bar{\Xi}_0 + \Xi_0 \equiv \bar{\theta}^+ + \theta^- \tag{5.13}$$

[10]The interaction terms in the continuum limit theory with the doublers included as separate "flavors" most likely violates Lorentz symmetry, as in [25]; we note, however, that the arguments of that paper are not directly applicable as the supersymmetry transformations are not the ones assumed there.

[11]We thank S. Catterall for making us think of twisted mass deformations.



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into an integral over the full superspace. (We note in passing that a more general combination is allowed: $Q_A^\delta = e^{i\delta} Q_- + e^{-i\delta}\bar{Q}_+$. However, it can be shown that up to a rescaling of parameters of the action by an overall phase, this is of no consequence to the continuum limit that is obtained. We therefore set $\delta = 0$ in all of our considerations below.)

Consider first the following nonlocal interaction term, invariant under $I$, $Z_{2F}$, and $\mathrm{U}(1)_A$, but violating $\mathrm{U}(1)_V$ and $Z_4$:

$$F_{\vec{m},\vec{n}} = \int d\theta^- d\bar{\theta}^+ \, (\bar{\theta}^+ + \theta^-) \left( W'(U_{\vec{m}})\, \Xi_{\vec{n}} + \bar{W}'(\bar{U}_{\vec{m}}) \bar{\Xi}_{\vec{n}} \right) = (F_1 + F_2)_{\vec{m},\vec{n}} \,, \tag{5.14}$$

where prime denotes the derivative of the function $W$ w.r.t. its argument. The term (5.14), as indicated above, can be split into two parts, according to their $Z_4$ properties, where:

$$F_{1,\vec{m},\vec{n}} = \int d\theta^- W'(U_{\vec{m}})\, \Xi_{\vec{n}}\Big|_{\bar{\theta}^+=0} - \int d\bar{\theta}^+ \bar{W}'(\bar{U}_{\vec{m}})\, \bar{\Xi}_{\vec{n}}\Big|_{\theta^-=0} \,, \tag{5.15}$$

is $Z_4$, as well as $I$, $\mathrm{U}(1)_A$, and $Z_{2F}$ invariant. If we recall now that $\Xi\big|_{\bar{\theta}^+=0} = \psi_+ + \theta^- F$ and the $I$ conjugate relation for $\bar{\Xi}$ implied by (5.2), we easily find that (5.15) contains precisely the interactions that form the usual $F$-terms in the continuum limit, with $W(U)$ — the superpotential.

The second part of $F$, called $F_2$ in (5.14), also respects $I$, $\mathrm{U}(1)_A$ and $Z_{2F}$, but violates the discrete $Z_4$ subgroup of euclidean rotations:

$$\begin{aligned} F_{2,\vec{m},\vec{n}} &= -\int d\bar{\theta}^+ W'(U_{\vec{m}})\, \Xi_{\vec{n}}\Big|_{\theta^-=0} + \int d\theta^- \bar{W}'(\bar{U}_{\vec{m}})\, \bar{\Xi}_{\vec{n}}\Big|_{\bar{\theta}^+=0} \\ &= iW'(U_{\vec{m}})\, \Delta_{\bar{z}} U_{\vec{n}}\Big|_{\theta^-=0} - i\bar{W}'(\bar{U}_{\vec{m}})\, \Delta_z \bar{U}_{\vec{n}}\Big|_{\bar{\theta}^+=0} \\ &= iW'(\phi_{\vec{m}})\, \Delta_{\bar{z}} \phi_{\vec{n}} - i\bar{W}'(\bar{\phi}_{\vec{m}})\, \Delta_z \bar{\phi}_{\vec{n}} \,. \end{aligned} \tag{5.16}$$

The lessons from eqs. (5.14)–(5.16) are that:

i.) It is possible to construct interactions that give rise to an $F$ term in the continuum limit and preserve an exact lattice supersymmetry.

ii.) The exact lattice supersymmetry of the $F$ terms requires that these interactions include the $Z_4$ violating term (5.16). However, also from (5.16), we see that, just like in quantum mechanics, if we consider the local action, $\sum_{\vec{m}} F_{\vec{m},\vec{m}}$, the $Z_4$ violating term $F_2$, is reduced to a total derivative in the continuum limit. As in quantum mechanics, we expect it to be irrelevant (if the ultraviolet behavior of the lattice theory is sufficiently soft).

To lift the doublers, we incorporate a nonlocal mass terms via the $F$ terms. Thus, the lattice superpotential interaction that we will consider is:

$$\begin{aligned} S_F = {} & a^2 \sum_{\vec{m}} \int d\theta^- d\bar{\theta}^+ (\bar{\Xi}_0 + \Xi_0) \times \\ & \times \left( ra\, U_{\vec{m}}\, \Delta^2\, \Xi_{\vec{m}} + r^* a\, \bar{U}_{\vec{m}}\, \Delta^2\, \bar{\Xi}_{\vec{m}} + \sum_{k\geq 2} g_k\, U_{\vec{m}}^{k-1}\, \Xi_{\vec{m}} + g_k^*\, \bar{U}_{\vec{m}}^{k-1}\, \bar{\Xi}_{\vec{m}} \right) . \end{aligned} \tag{5.17}$$

Here $r, r^*$ are the complex Wilson term coefficients, while $g_k, k \geq 2$ are the complex superpotential couplings of unit mass dimension. The local part of the superpotential, implicit in eq. (5.17), is:

$$W(U) = \sum_{k \geq 2} \frac{g_k}{k} U^k . \tag{5.18}$$

The laplacian $\Delta^2$, which appears in the Wilson term, is given explicitly in (D.2). The full component expression of the $F$-term lagrangian is easily found, generalizing to the multifield case, to be:

$$\begin{aligned} S_F = a^2 \sum_{\vec{m}} \Big\{ & W_{IJ} \, \psi^I_{-,\vec{m}} \psi^J_{+,\vec{m}} + W_I F^I_{\vec{m}} + i W_I \Delta_{\bar{z}} \phi^I_{\vec{m}} + \\ & + \bar{W}_{\bar{I}\bar{J}} \, \bar{\psi}^{\bar{I}}_{+,\vec{m}} \bar{\psi}^{\bar{J}}_{-,\vec{m}} + \bar{W}_{\bar{I}} \bar{F}^{\bar{I}}_{\vec{m}} - i \bar{W}_{\bar{I}} \Delta_z \bar{\phi}^{\bar{I}}_{\vec{m}} + \\ & + r a \, \psi^I_{-,\vec{m}} \Delta^2 \psi^I_{+,\vec{m}} + r a \, \phi^I_{\vec{m}} \Delta^2 F^I_{\vec{m}} + \\ & + r^* a \, \bar{\psi}^{\bar{I}}_{+,\vec{m}} \Delta^2 \bar{\psi}^{\bar{I}}_{-,\vec{m}} + r^* a \, \bar{\phi}^{\bar{I}}_{\vec{m}} \Delta^2 \bar{F}^{\bar{I}}_{\vec{m}} \Big\} . \end{aligned} \tag{5.19}$$

(In appendix D, we demonstrate how to incorporate the nonlocal Wilson terms into the superpotential by an appropriate modification of (5.18).) We note that the three-derivative bilinear scalar terms, of the form $i\phi_{\vec{m}} \Delta^2 \Delta_{\bar{z}} \phi_{\vec{m}}$, which would seem to appear in the Wilson term from (5.17) vanish after summation over the periodic lattice. As usual, $W_I, W_{IJ}$ denote derivatives of the superpotential (5.18) with respect to the fields $U^I$, evaluated at $\phi_{\vec{m}}$. The naive continuum limit of the lattice action is, by inspection of (5.19) the usual superpotential interaction in the $(2, 2)$ continuum theory. The irrelevant nonlocal mass terms, as we show in appendix D, lift the doublers by giving them mass of order $|r|a^{-1}$ and ensure that the spectrum of the lattice theory matches that of the continuum.

It is important to note that the exact lattice supersymmetry and the other global symmetries do not require the action to have the form (5.19) (with $D$-term (5.9)). In fact, if we were to ask for the most general involution, fermion number, $\mathrm{U}(1)_A$, and $Q_A$ invariant action of lowest dimension (i.e. local and without superspace derivatives on the fields), we would find the following relevant local term:

$$G_{\vec{m}} = \int d\theta^- d\bar{\theta}^+ (\bar{\theta}^+ + \theta^-) \left( \mathcal{G}(U_{\vec{m}}, \bar{U}_{\vec{m}}) \, \Xi_{\vec{m}} + \mathcal{G}^*(\bar{U}_{\vec{m}}, U_{\vec{m}}) \, \bar{\Xi}_{\vec{m}} \right) , \tag{5.20}$$

where $\mathcal{G}$ is now an arbitrary function of $U$ *and* $\bar{U}$ — as opposed to $W'(U)$ of (5.14) — of unit mass dimension. An additional marginal term would have form similar to (5.20), but $\mathcal{G}$ would be then linear in derivatives. Consider now the structure of (5.20) in more detail. As in the case of (5.14), we can decompose $G = G_1 + G_2$, where $G_1$ preserves the discrete rotation $Z_4$ symmetry, while $G_2$ does not:

$$\begin{aligned} G_1 &= \int d\theta^- \mathcal{G}(U, \bar{U}) \Xi \big|_{\bar{\theta}^+ = 0} + \text{I.c.} = \mathcal{G}(\phi, \bar{\phi}) \, \psi_- \psi_+ + \mathcal{G}_{,\phi}(\phi, \bar{\phi}) \, F + \text{I.c.} \\ G_2 &= -\int d\bar{\theta}^+ \mathcal{G}(U, \bar{U}) \Xi \big|_{\theta^- = 0} + \text{I.c.} = i\mathcal{G}(\phi, \bar{\phi}) \, \Delta_{\bar{z}} \phi + \mathcal{G}_{,\bar{\phi}}(\phi, \bar{\phi}) \, \bar{\psi}_+ \psi_+ + \text{I.c.} . \end{aligned} \tag{5.21}$$

Thus, $G_1$ contains nonholomorphic corrections to the superpotential, while $G_2$ consists of $Z_4$ violating terms, related to the nonholomorphic $F$ terms in $G_1$ by the exact lattice supersymmetry. Thus, the symmetries of the lattice action are not sufficient to guarantee that the $(2,2)$ supersymmetric continuum limit is achieved. Moreover, there are no additional symmetries, within the present approach, that can forbid terms like (5.20) but allow (5.17). We conclude that the action with the desired $(2,2)$ supersymmetric continuum limit is not generic.

We thus need to address the following question: given a supersymmetric lattice theory with action given by (5.9), and with superpotential (5.17), are nonholomorphic, non-Lorentz-invariant terms in the action of the form (5.21) generated quantum mechanically?

In this paper, we will only address the case of Landau-Ginzburg (or Wess-Zumino) models. In other words, we consider supersymmetric lattice theories with flat Kähler metric, $K_{I\bar{J}} = \delta_{I\bar{J}}$, and some polynomial superpotential. We hope to address the (significantly more involved) renormalization of models with non-trivial Kähler metric, both with supersymmetric and non-supersymmetric Wilson terms in the future.

To study the quantum continuum limit, we consider in some detail the counting of divergences in lattice perturbation theory in the Landau-Ginzburg model, in appendix D. As discussed there, the lattice power counting rules show that the lattice introduces several extra divergent graphs (compared to the continuum case) due to the higher-derivative vertices induced by the supersymmetrization of the Wilson term and by the interaction term (5.22). However, as also shown in appendix D, the exact lattice supersymmetry is sufficient to ensure that all divergent as $a \to 0$ lattice graphs cancel and that the lattice theory is finite. That is, the net sum of all lattice perturbation theory Feynman diagrams contributing to any proper vertex can be seen to have a negative degree of divergence (determined by lattice power-counting), also due to the exact lattice supersymmetry. This ensures that the lattice and continuum perturbation expansions are identical in the $a \to 0$ limit[12] and that the continuum $(2,2)$ supersymmetric continuum limit is achieved without any fine tuning. We note that the finiteness due to the exact lattice supersymmetry was crucial to the argument.

Finally, we note that the $Z_4$ violating term in the lattice action (5.19), despite appearing relevant as written, is, in fact, an irrelevant operator. This can be seen, again, by using periodicity of the lattice. In the $W(\phi) = g\phi^3/3$ case (quadratic terms in the superpotential do not contribute to the $Z_4$ violating term due to periodicity on the lattice), the $Z_4$-violating term in the action can be identically written as:

$$a^2 \sum_{\vec{m}} g\phi_{\vec{m}}^2 \Delta_{\bar{z}} \phi_{\vec{m}} \equiv \frac{1}{6}\, a^2 \sum_{\vec{m}} \, a^2 g \left( (\Delta_1^+ \phi_{\vec{m}})^3 - i(\Delta_2^+ \phi_{\vec{m}})^3 \right) , \tag{5.22}$$

with $\Delta_i^+$ — the forward derivative in the $i$-th direction and we used symmetric differences in $\Delta_{\bar{z}}$. It is easy to check that the $Z_4$ violating term in the lattice action is of order $a^2$ for any power $k > 2$ in the superpotential, but the identity is not as simple as for the $k = 3$

[12]For general theorems on asymptotic expansions of lattice Feynman integrals and lattice power counting rules, see [26].

case above. The irrelevant Lorentz violating term has no effect on the continuum theory due to the finiteness of the lattice theory (another way to see that the nonholomorphic terms of (5.21) are irrelevant, as they are related to the Lorentz violating ones by the lattice supersymmetry).

Thus, as mentioned in section 1.5, the results of this section demonstrate that the $(2,2)$ LG models can be simulated on the lattice.

## Acknowledgments

We would like to thank Kentaro Hori and Simon Catterall for useful discussions. This work was supported by the National Science and Engineering Research Council of Canada and the Ontario Premier's Research Excellence Award.

## A. The transfer matrix and Witten index of the lattice supersymmetric quantum mechanics

Consider the lattice partition function (2.19) after a convenient change of fermionic variables $\psi^i = \bar\eta^{i+1}$, $\chi^i = \eta^i$:

$$Z_N = c^N \prod_{i=1}^{N} \int d\bar\eta^i d\eta^i \int_{-\infty}^{\infty} dx^i \, e^{-S} \,, \tag{A.1}$$

with the action (2.18) in terms of the new variables:

$$S = \sum_{i=1}^{N} \left[ \frac{a}{2} \left( \frac{x^{i+1} - x^i}{a} + h'(x^{i+1}) \right)^2 - \bar\eta^{i+1}\eta^i \left( a h''(x^i) + 1 \right) + \bar\eta^i \eta^i \right] . \tag{A.2}$$

The lattice action above is not reflection positive. Nevertheless, we will show that in the continuum limit it defines a hermitean hamiltonian acting on a positive norm Hilbert space — the hamiltonian of supersymmetric quantum mechanics. The lack of reflection positivity comes from the cross term in the expansion of the square in the bosonic action (which would be a total derivative, $\frac{\partial}{\partial t} h$ in the continuum limit). The presence of this term is required by the exact lattice supersymmetry of the action. It is natural to expect that in the continuum limit its effect will be irrelevant; this is what we want to demonstrate here.[13]

To construct the transfer matrix and hamiltonian, see [27], we first introduce, at each time slice, a Hilbert space which is a tensor product of a bosonic and fermionic space. The bosonic Hilbert space is that of square integrable functions on the line. We use the basis of position eigenstates, $\{|x\rangle, \langle x'|x\rangle = \delta(x'-x)\}$, where the momentum and position operators, $[\hat p, \hat q] = -i$, act as $\hat q|x\rangle = |x\rangle x$ and $e^{i\hat p \Delta}|x\rangle = |x+\Delta\rangle$ (note that we continue using

[13] As already mentioned in section 2, the choice of discretization of the derivative in (A.2) corresponds to a Wilson term with $r \equiv 1$. This is the choice where only nearest-neighbor interactions occur, for which the construction of the transfer matrix and hamiltonian from the euclidean lattice action is most straightforward.

the dimensions of section 2: $x$ has mass dimension $-1/2$, $a$ has dimension of length, while the superpotential $h(x)$ is dimensionless). The fermionic Hilbert space is two dimensional and is spanned by the vectors $|0\rangle$ and $|1\rangle$. The fermionic creation and annihilation operators obey $\{\hat{b}^\dagger, \hat{b}\} = 1$, such that $\hat{b}|0\rangle = 0$, $|1\rangle = \hat{b}^\dagger|0\rangle$. The fermionic coherent states are defined as $|\eta\rangle \equiv |0\rangle + |1\rangle\eta$, $\langle\eta| = \langle 0| + \bar{\eta}\langle 1|$, where $\eta$ and $\bar{\eta}$ are Grassmann variables. We then recall the usual relations for the decomposition of unity, $\langle\eta'|\eta\rangle = e^{\bar{\eta}'\eta}$; $\hat{1} = \int d\bar{\eta}d\eta e^{-\bar{\eta}\eta}|\eta\rangle\langle\eta|$, and for traces[14] of operators $\mathcal{O}$ on the fermionic Hilbert space: $\mathrm{Tr}\,\mathcal{O} = \int d\bar{\eta}d\eta e^{-\bar{\eta}\eta}\langle\eta|\mathcal{O}| - \eta\rangle$; $\mathrm{Tr}(-1)^F\mathcal{O} = \int d\bar{\eta}d\eta e^{-\bar{\eta}\eta}\langle\eta|\mathcal{O}|\eta\rangle$, with $(-1)^F|0\rangle = |0\rangle$. We then define the transfer matrix by the equality:

$$
\begin{aligned}
Z_N &= c^N \prod_{i=1}^{N} \int d\bar{\eta}^i d\eta^i \int_{-\infty}^{\infty} dx^i \, e^{-S} \equiv \mathrm{Tr}(-1)^F \, \hat{T}^N \\
&= \prod_{i=1}^{N} \int d\bar{\eta}^i d\eta^i e^{-\bar{\eta}^i\eta^i} \int_{-\infty}^{\infty} dx^i \, \times \\
&\quad \times \langle\eta^N, x^N|\hat{T}|\eta^{N-1}, x^{N-1}\rangle\langle\eta^{N-1}, x^{N-1}|\hat{T}|\eta^{N-2}, x^{N-2}\rangle \times \cdots \times \\
&\quad \times \langle\eta^2, x^2|\hat{T}|\eta^1, x^1\rangle\langle\eta^1, x^1|\hat{T}|\eta^N, x^N\rangle\,,
\end{aligned}
\tag{A.3}
$$

or, equivalently, through its matrix elements:

$$
\langle\eta^{i+1}, x^{i+1}|\hat{T}|\eta^i, x^i\rangle = c \exp\left[ -\frac{a}{2}\left(\frac{x^{i+1} - x^i}{a} + h'(x^{i+1})\right)^2 + \bar{\eta}^{i+1}\eta^i(1 + ah''(x^i))\right]. \tag{A.4}
$$

Using $\langle\eta'|1 - X\hat{b}^\dagger\hat{b}|\eta\rangle = e^{(1-X)\bar{\eta}'\eta}$, it is straightforward to check that the $\hat{T}$ operator with matrix elements (A.4) is given by:

$$
\hat{T} = c\int_{-\infty}^{\infty} dz \, \exp\left(-\frac{a}{2}\left(\frac{z}{a} + h'(\hat{q})\right)^2\right) \exp\left(iz\hat{p}\right) \left(1 + ah''(\hat{q})\,\hat{b}^\dagger\hat{b}\right)\,. \tag{A.5}
$$

In the small-$a$ limit, this operator becomes, with $c = (2\pi a)^{-\frac{1}{2}}$:

$$
\begin{aligned}
\hat{T} &= \exp\left(-\frac{a}{2}\hat{p}^2 - \frac{a}{2}i\left(h'(\hat{q})\,\hat{p} + \hat{p}\,h'(\hat{q})\right) + ah''(\hat{q})\,\left(\hat{b}^\dagger\hat{b} - \frac{1}{2}\right)\right) e^{\mathcal{O}(a^2)} \\
&= e^{h(\hat{q})} \exp\left(-a\left(\frac{\hat{p}^2}{2} + \frac{h'(\hat{q})^2}{2} - \frac{h''(\hat{q})}{2}\left[\hat{b}^\dagger, \hat{b}\right]\right)\right) e^{-h(\hat{q})}\, e^{\mathcal{O}(a^2)}\,.
\end{aligned}
\tag{A.6}
$$

The term in the middle exponent, proportional to a single power of the lattice spacing, is easily recognized as the hamiltonian of the supersymmetric quantum mechanics, $H_{SQM}$. The limit of small lattice spacing should, of course, be understood in the weak sense (as for arbitrarily small $a$ there always exist large enough $x$ such that the order $a^2$ term is

[14] We note in passing that the first of the two relations that follow can, along with the expression for the $T$-matrix (A.5), be used to give a path integral representation of the finite temperature partition function as well. Clearly, this amounts to switching the sign of $\eta$ only in the second term of eq. (A.2) (equivalently, the second term of (A.4)) at a single lattice site only (despite appearances, eq. (A.3) shows that discrete time translation invariance is not broken, since the point can be freely moved around). The exact supersymmetry is then globally broken by the boundary conditions, but locally present.



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important; however, for a potential sufficiently strong at infinity these values of $x$ make an exponentially small contribution to the path integral). As eq. (A.6) shows, in the $a \to 0$ limit, the $\hat{T}$ matrix (A.5), with matrix elements (A.4), is conjugate to $\hat{T}_{cont.} \equiv e^{-aH_{SQM}}$:

$$\hat{T} \simeq e^{h(\hat{q})}\, e^{-aH_{SQM}}\, e^{-h(\hat{q})}\,. \tag{A.7}$$

Inserting (A.7) into (A.3), one observes that the $e^{\pm h}$ factors cancel out of the partition function, and we are left with the usual "naive" path integral representation for the partition function.

We thus conclude that, in the continuum limit the two discretizations — the "naive" and the supersymmetric, with $\hat{T}$ of eq. (A.4), are equivalent: both converge to $Z = \mathrm{Tr}(-1)^F e^{-\beta H_{SQM}}$. The supersymmetric discretization, however, enjoys nice properties already at finite $N$ and $a$; for example, as we show below, it gives the correct value of the Witten index already at finite lattice spacing. Moreover, if one was interested in numerical computations of correlation functions in supersymmetric quantum mechanics, one would find much faster convergence to the supersymmetric continuum limit in the case of a supersymmetric discretization.

We now elaborate on the property of the supersymmetric lattice action alluded to above, and show how the exact lattice supersymmetry of (2.19) (or of (A.1)) implies that the correct value for the Witten index is obtained already at finite lattice spacing. To this end we take (2.19) in the form:

$$Z_N = (2\pi a)^{-\frac{N}{2}} \int \prod_{i=1}^{N} dx^i d\chi^i d\psi^i \exp\left(-\frac{a}{2}\mathcal{N}^i\mathcal{N}^i(x) - a\,\chi^i \frac{\partial \mathcal{N}^i(x)}{\partial x^j}\psi^j\right) \tag{A.8}$$

where $\mathcal{N}^i(x) \equiv h'(x^i) + (x^i - x^{i-1})/a$ is the Nicolai variable (2.21) (summation over repeated indices is understood in this discussion). Both the measure and action are invariant under the nilpotent supersymmetry generated by $Q$, which acts as implied by (2.12): $\delta_\epsilon x^i = \epsilon\psi^i$, $\delta_\epsilon\psi^i = 0$, $\delta_\epsilon\chi^i = -\epsilon\mathcal{N}^i$. $Q$-invariance of the measure and action imply that (schematically) $\int dx d\chi d\psi\, e^{-S}\, \delta_\epsilon X(x,\chi,\psi) = 0$, i.e., that correlation functions of $Q$-exact operators vanish.

Consider now the $Q$-variation of a particular $X(x,\chi,\psi)$, chosen as $X = -\chi^i g^i(x)$, where $g^i$ is some function of the $x^k$: $\delta_\epsilon X = \epsilon(\mathcal{N}^i g^i + \chi^i \frac{\partial g^i}{\partial x_j}\psi^j)$. The crucial point is that this $Q$-variation of $X$ is the same as the variation of the action, $S = \frac{1}{2}(\mathcal{N}^i(x))^2 + \chi^i\, \frac{\partial \mathcal{N}^i(x)}{\partial x^j}\,\psi^j$, under an $x$-dependent shift of the Nicolai variable: $\delta_g \mathcal{N}^i = g^i(x)$: $\epsilon\delta_g S = \delta_\epsilon X$. Therefore, under a shift of the Nicolai variable, the change of $Z_N$ is a correlation function of a $Q$-exact operator. Since such correlators vanish, the conclusion of the previous paragraph implies that $Z_N$ is invariant under deformations of the Nicolai variable (provided they do not change the asymptotics of $h$ at infinity). In particular, upon choosing $g^i = -(x^i - x^{i-1})/a$, the lattice sites decouple and we obtain a simple expression for $Z_N$:

$$\begin{aligned} Z_N &= \left(\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi a}} \int d\chi d\psi\, \exp\left(-\frac{a}{2}\, h'(x)^2 - a h''(x)\,\chi\psi\right)\right)^N \\ &= \left(\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}}\, h''(x) \exp\left(-\frac{1}{2}\, h'(x)^2\right)\right)^N = \left(\sum_{x^*} \frac{h''(x^*)}{|h''(x^*)|}\right)^N, \end{aligned} \tag{A.9}$$

where the sum over the critical points $x^*$ of the superpotential $h(x)$, $\sum_{x^*} \frac{h''(x^*)}{|h''(x^*)|}$, takes values $\pm 1$ or 0, depending on whether the superpotential is "odd" or "even" at infinity. This sum, in fact, equals the Witten index of the continuum quantum mechanics with superpotential $h$. Thus, the supersymmetric lattice path integral precisely reproduces, already at finite $N$, the continuum value of the Witten index, for odd total number of lattice points $N$. This is to be expected: recall that in the continuum one calculates the index by compactifying on a euclidean circle with periodic boundary conditions. One then calculates the determinant of the quantum fluctuations around field configurations on which the path integral is localized ($\dot{x} = h'(x) = 0$). A complete cancellation between the nonzero modes occurs only if an odd (of course, infinite) number of bosonic modes is present (a zero mode and an even number of Kaluza-Klein modes). In our lattice regularization this corresponds to having an odd number of lattice sites.

## B. A-type lattice superfield kinematics

In the standard superspace notation, the supersymmetry generators in (5.1) are represented as differential operators acting on superfields:

$$\begin{aligned} Q_{\pm} &= \frac{\partial}{\partial \theta^{\pm}} + i\bar{\theta}^{\pm}\partial_{\pm} \,, \\ \bar{Q}_{\pm} &= -\frac{\partial}{\partial \bar{\theta}^{\pm}} - i\theta^{\pm}\partial_{\pm} \,. \end{aligned} \tag{B.1}$$

A supersymmetry transformation with parameters $\epsilon^{\pm}$, $\bar{\epsilon}^{\pm}$ is generated by the action of:

$$\hat{\delta} = \epsilon^{-}Q_{-} + \epsilon^{+}Q_{+} - \bar{\epsilon}^{-}\bar{Q}_{-} - \bar{\epsilon}^{+}\bar{Q}_{+} \tag{B.2}$$

on superfields. Supersymmetric actions are written as integrals over superspace of various superfields. The variation of a superspace lagrangian density is found by acting with the differential operator $\delta$. Since spacetime derivatives appear in the $Q$'s, the lagrangian densities are $\delta$-invariant only up to total derivative terms. The validity of the Leibnitz rule for spacetime derivatives is thus crucial for preserving supersymmetry; as stated many times above, this is the main obstacle for preserving the supersymmetry algebra on the lattice (in interacting models). The appearance of the "troublesome" spacetime derivatives in (B.1) can, in its turn, be traced to the non-anticommutativity of the supercharges. Now, recall that the supersymmetry generators can also be written in the following form:

$$\begin{aligned} Q_{+} &= e^{-X_{+}}\frac{\partial}{\partial \theta^{+}}e^{X_{+}} \equiv \frac{\partial}{\partial \theta^{+}} + \left(\frac{\partial}{\partial \theta^{+}}X_{+}\right), \qquad \text{where } X_{+} \equiv i\theta^{+}\bar{\theta}^{+}\partial_{+} \,, \\ Q_{-} &= e^{-X_{-}}\frac{\partial}{\partial \theta^{-}}e^{X_{-}} \equiv \frac{\partial}{\partial \theta^{-}} + \left(\frac{\partial}{\partial \theta^{-}}X_{-}\right), \qquad \text{where } X_{-} \equiv i\theta^{-}\bar{\theta}^{-}\partial_{-} \,, \\ \bar{Q}_{+} &= -e^{X_{+}}\frac{\partial}{\partial \bar{\theta}^{+}}e^{-X_{+}} \equiv -\frac{\partial}{\partial \bar{\theta}^{+}} + \left(\frac{\partial}{\partial \bar{\theta}^{+}}X_{+}\right), \\ \bar{Q}_{-} &= -e^{X_{-}}\frac{\partial}{\partial \bar{\theta}^{-}}e^{-X_{-}} \equiv -\frac{\partial}{\partial \bar{\theta}^{-}} + \left(\frac{\partial}{\partial \bar{\theta}^{-}}X_{-}\right). \end{aligned} \tag{B.3}$$

It follows from the above equation that if two supercharges anticommute, with an appropriate change of coordinates, the corresponding differential operators can be represented simply as derivatives with respect to the odd superspace coordinates. For example, it follows from eq. (B.3) that the anticommuting operators $\bar{Q}_+$ and $\bar{Q}_-$ (our "$B$-type") can be represented solely by supercoordinate derivatives upon conjugation:

$$\bar{Q}_\pm = e^{X_+ + X_-}\,\bar{Q}^B_\pm\, e^{-X_+ - X_-}\,, \qquad \bar{Q}^B_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm}\,. \tag{B.4}$$

We can also see from (B.3) that supersymmetry generators whose anticommutator involves a spatial derivative can not be conjugated to purely supercoordinate derivative — for example, representing $Q_+$ and $\bar{Q}_+$ by pure $\theta$ derivatives requires opposite conjugations, with $e^{X_+}$ and $e^{-X_+}$, respectively. For further use, we note that the supercovariant derivatives can be written similar to (B.3):

$$\begin{aligned} D_+ &= e^{X_+}\,\tfrac{\partial}{\partial\theta^+}\,e^{-X_+}\,, & D_- &= e^{X_-}\,\tfrac{\partial}{\partial\theta^+}\,e^{-X_-} \\ \bar{D}_+ &= -e^{-X_+}\tfrac{\partial}{\partial\bar{\theta}^+}e^{X_+}\,, & \bar{D}_- &= -e^{-X_-}\,\tfrac{\partial}{\partial\bar{\theta}^-}e^{X_{--}}\,. \end{aligned}$$

Clearly, for the "$A$-type," a simultaneous representation of the $\bar{Q}_+$ and $Q_-$ as purely supercoordinate derivatives also exists:

$$\begin{aligned} Q_- &= e^{X_+ - X_-}\,Q^A_-\,e^{X_- - X_+}\,, &\quad \text{with } Q^A_- &\equiv \frac{\partial}{\partial\theta^-}\,, \\ \bar{Q}_+ &= e^{X_+ - X_-}\,\bar{Q}^A_+\,e^{X_- - X_+}\,, &\quad \text{with } \bar{Q}^A_+ &\equiv -\frac{\partial}{\partial\bar{\theta}^+}\,. \end{aligned} \tag{B.6}$$

The form of the remaining $Q_+$ and $\bar{Q}_-$ after conjugation with $e^{X_- - X_+}$ can easily be worked out and seen to involve spacetime derivatives. Thus, these two remaining supercharges will not be exact symmetries of an interacting $A$-type lattice action.

Our next task is to construct the type-$A$ lattice superfields. As we are interested in theories whose continuum limit is a theory of chiral superfields, we begin by transforming the familiar continuum chiral superfields into the basis where $Q^-$ and $\bar{Q}^+$ act by shifts of their respective odd superspace coordinates.

A chiral superfield $\Phi$ obeys the covariant constraint $\bar{D}_\pm\Phi = 0$. This constraint is easy to solve in the chiral basis, defined by $Q = e^{-X_- - X_+}\,Q^\chi\,e^{X_- + X_+}$, where $Q_\pm$ and $\bar{D}_\pm$ are simultaneously represented in terms of pure $\theta$ derivatives. The chiral superfield's component expansion is then easily seen to be:

$$\begin{aligned} \Phi(x^\pm,\theta^\pm,\bar{\theta}^\pm) &= e^{-X_+ - X_-}\,\Phi^\chi(x^\pm,\theta^\pm)\,e^{X_+ + X_-} \\ &= \phi(y^\pm) + \theta^+\psi_+(y^\pm) + \theta^-\psi_-(y^\pm) + \theta^+\theta^- F(y^\pm)\,. \end{aligned} \tag{B.7}$$

In (B.7) we denote by $y^\pm$ the coordinate $y^\pm = x^\pm - i\theta^\pm\bar{\theta}^\pm$. To transform to the $A$-type basis we conjugate the fields and charges as $\Phi^A = e^{X_- - X_+}\,\Phi(x^\pm,\theta^\pm,\bar{\theta}^\pm)\,e^{X_+ - X_-}$. Thus, the $A$-basis conjugated chiral superfield is:

$$\begin{aligned} &\Phi^A\,(x^+,x^-,\theta^\pm,\bar{\theta}^\pm) = \\ &\qquad = e^{X_- - X_+}\,\Phi(x^\pm,\theta^\pm,\bar{\theta}^\pm)\,e^{X_+ - X_-} \end{aligned}$$

$$
\begin{aligned}
&= e^{X_- - X_+}\left[\phi(y^\pm) + \theta^+\psi_+(y^\pm) + \theta^-\psi_-(y^\pm) + \theta^+\theta^- F(y^\pm)\right] e^{X_+ - X_-} \\
&= \phi(x^+ - 2i\theta^+\bar\theta^+, x^-) + \theta^+\psi_+(x^+ - 2i\theta^+\bar\theta^+, x^-) \\
&\quad + \theta^-\psi_-(x^+ - 2i\theta^+\bar\theta^+, x^-) + \theta^+\theta^- F(x^+ - 2i\theta^+\bar\theta^+, x^-) + \\
&= \phi - 2i\theta^+\bar\theta^+\partial_+\phi + \theta^+\psi_+ + \theta^-\psi_- - 2i\theta^-\theta^+\bar\theta^+\partial_+\psi_- + \theta^+\theta^- F\,,
\end{aligned}
\tag{B.8}
$$

where all component fields in the last line are functions of $x^\pm$. In the $A$-representation above, the $Q^-$ and $\bar{Q}^+$ supercharges now act simply by shifting $\theta^-$ and $\bar\theta^+$. We can rewrite the chiral superfield (B.8) as follows:

$$
\Phi^A = \left(\phi + \theta^-\psi_-\right) + \theta^+\left(\psi_+ - 2i\bar\theta^+\partial_+\phi + \theta^- F + 2i\theta^-\bar\theta^+\partial_+\psi_-\right) \equiv U + \theta^+\Xi\,. \tag{B.9}
$$

The two terms in brackets in (B.9) are $\theta^-, \bar\theta^+$-dependent superfields, with irreducible action of the $A$-type supercharges. Thus, under the action of $Q^-$ and $\bar{Q}^+$ alone, the chiral superfield $\Phi$ splits into two irreducible components — the $U$ and $\Xi_+$ superfields:

$$
\begin{aligned}
U(x^\pm, \theta^-) &= \phi + \theta^-\psi_- \\
\Xi(x^\pm, \theta^-, \bar\theta^+) &= \psi_+ - 2i\bar\theta^+\partial_+\phi + \theta^- F + 2i\theta^-\bar\theta^+\partial_+\psi_-\,.
\end{aligned}
\tag{B.10}
$$

Similarly, since an antichiral superfield can be written as $\bar\Phi(x^\pm, \theta^\pm, \bar\theta^\pm) = \bar\phi(\bar{y}^\pm) - \bar\theta^-\bar\psi_-(\bar{y}^\pm) - \bar\theta^+\bar\psi_+(\bar{y}^\pm) + \bar\theta^-\bar\theta^+\bar{F}(\bar{y}^\pm)$, with $\bar{y}^\pm = x^\pm + i\theta^\pm\bar\theta^\pm$, it follows that an antichiral superfield in the $A$ basis is given by:

$$
\begin{aligned}
\bar\Phi^A &= e^{X_- - X_+}\,\bar\Phi(x^\pm, \theta^\pm, \bar\theta^\pm) e^{X_+ - X_-} = \bar\phi(x^+, x^- + 2i\theta^-\bar\theta^-) + \cdots \\
&= \left(\bar\phi - \bar\theta^+\bar\psi_+\right) - \bar\theta^-\left(\bar\psi_- + 2i\theta^-\partial_-\bar\phi - \bar\theta^+\bar{F} - 2i\theta^-\bar\theta^+\partial_-\bar\psi_+\right) \equiv \bar{U} - \bar\theta^-\bar\Xi\,.
\end{aligned}
\tag{B.11}
$$

We see that the antichiral field $\bar\Phi$ also splits in to two superfields that transform irreducibly under the type-$A$ supercharges. The superfields $\bar{U}$ and $\bar\Xi$ are defined by eq. (B.11). Their properties under I-conjugation were discussed (for their lattice version) in section 5.1 above; there it was seen that this conjugation is not the usual Minkowski space complex conjugatation of $U$ and $\Xi$.

As explained in the introduction, our lattice theory (prior to lifting spectrum doublers) will preserve exactly the supersymmetries represented by shifts of $\theta^-$ and $\bar\theta^+$. We will construct $Q^-$ and $\bar{Q}^+$ invariants using a lattice version of the superfields $U, \bar{U}$, $\Xi, \bar\Xi$. Since the action of these supersymmetries does not involve any spacetime derivatives, the latticization, including interactions, will not be in conflict with the $Q^-$ and $\bar{Q}^+$ supersymmetries.

We now introduce two pairs of bosonic $(U, \bar{U})$ and fermionic $(\Xi, \bar\Xi)$ lattice superfields by simply latticizing their continuum definitions (B.11) and (B.10). Before introducing the lattice, we rotate to euclidean space, by replacing $x^0$ by $-ix^2$. Thus, $\partial_0 = i\partial_2$. We also introduce the complex notation $z = x^1 + ix^2$, $\partial_z = (\partial_1 - i\partial_2)/2$. Since $\partial_\pm = (\partial_0 \pm \partial_1)/2$, we find that $\partial_+ = \partial_{\bar{z}}$, $\partial_- = -\partial_z$.

We let our lattice have sites labeled by integer valued vectors $\vec{m}$, $m_{1,2} = 1, \ldots, N$, with $m_i + N \equiv m_i$. The continuum derivatives in the $i$-th direction are replaced by lattice derivatives. The symmetries and conjugation rules of the euclidean formulation will be discussed shortly; here we only note that any choice of derivative is compatible with the symmetries (5.7). Thus, until we make a choice of lattice derivatives, we denote the derivatives in the two directions by $\Delta_1$ and $\Delta_2$. We also denote $\Delta_z = \Delta_1 - i\Delta_2$ and $\Delta_{\bar z} = \Delta_1 + i\Delta_2$ (without factors of 1/2), so that the discrete form of $2\partial_+$ is $\Delta_{\bar z}$ and that of $2\partial_-$ is $-\Delta_z$. In fact, making these replacements in (B.10) and their I-conjugates, we arrive at the lattice superfields (also given in eq. (5.2)):

$$\begin{aligned}
U_{\vec m} &= \phi_{\vec m} + \theta^- \psi_{-,\vec m}\,, \\
\Xi_{\vec m} &= \psi_{+,\vec m} - \bar\theta^+ i\Delta_{\bar z}\phi_{\vec m} + \theta^- F_{\vec m} + i\theta^-\bar\theta^+\Delta_{\bar z}\psi_{-,\vec m}\,, \\
\bar U_{\vec m} &= \bar\phi_{\vec m} - \bar\theta^+ \bar\psi_{+,\vec m}\,, \\
\bar\Xi_{\vec m} &= \bar\psi_{-,\vec m} - \theta^- i\Delta_z\bar\phi_{\vec m} - \bar\theta^+ \bar F_{\vec m} + i\theta^-\bar\theta^+\Delta_z\bar\psi_{+,\vec m}\,.
\end{aligned} \tag{B.12}$$

The $Q_-$ and $\bar Q_+$ transformations of (B.12) are easily read off by noting that they are simply given by shifts of $\theta^-$ and $\bar\theta^+$ as implied by (B.2) and (B.6). The transformations under the rest of the supercharges are easily worked out starting from eq. (B.2) and paying attention to the change of basis, with the result, denoting $\delta' = \epsilon^+ Q_+ - \bar\epsilon^- \bar Q_-$:

$$\begin{aligned}
\delta'\phi_{\vec m} &= \epsilon^+\psi_{+,\vec m}\,, & \delta'\psi_{-,\vec m} &= -\epsilon^+ F_{\vec m} - i\,\bar\epsilon^-\Delta_z\phi_{\vec m}\,, \\
\delta'\psi_{+,\vec m} &= 0\,, & \delta' F_{\vec m} &= -i\,\bar\epsilon^-\Delta_z\psi_{+,\vec m}\,.
\end{aligned} \tag{B.13}$$

The $\delta'$ transformations on the components if $\bar U$ and $\bar\Xi$ can be found by an $I$-conjugation of (B.13), using the rules of (5.6) and, $(\epsilon^+)^I = i\bar\epsilon^-$, $(\bar\epsilon^-)^I = i\epsilon^+$. It is also easily checked that the transformations generated by $Q_\pm$ and $\bar Q_\pm$ defined above obey a discretized version of the continuum algebra with $2\partial_+$ replaced by $\Delta_{\bar z}$ and $2\partial_-$ by $-\Delta_z$; and that the free action, including the $F$-type Wilson term (discussed in section 5.3) is invariant under all four supersymmetries, as in the 1d case.

## C. Twisted mass and Wilson terms

In this section, we continue our study of the nonlinear sigma model. In order to save space, we will only consider the case of the $CP^1$ model, as we believe that this case is sufficiently general to allow us to make all our points; needless to say, our results are more generally valid.

In projective coordinates, the bosonic part of the $CP^1$ model is described by a single complex scalar field $\phi$ with Kähler metric $K_{\phi\bar\phi} \sim (1 + \bar\phi\phi)^{-2}$ (in this section, we will not worry about normalization unless absolutely necessary). In the supersymmetric case, $\phi$ is the lowest component of a chiral superfield. In its supersymmetric lattice version the model is described by a single set of lattice superfields $U, \Xi$, and their I-conjugates, with component action (5.11), and with $K_{\phi\bar\phi}$ substituted for the general $K_{I\bar J}$, i.e.:

$$S_{CP^1} = a^2 \sum_{\vec m} \int d\bar\theta^+ d\theta^- \frac{\bar\Xi_{\vec m}\Xi_{\vec m}}{(1 + \bar U_{\vec m}\, U_{\vec m})^2}\,. \tag{C.1}$$

The continuum $CP^1$ model has three isometries, representing the action of the SO(3) global symmetry on $S^2 \simeq CP^1$ in projective coordinates. They are generated by the three holomorphic Killing vectors:[15]

$$X^1(\phi) = -\frac{i}{2}(1-\phi^2)\,, \qquad X^2(\phi) = \frac{1}{2}(1+\phi^2)\,, \qquad X^3(\phi) = -i\phi\,, \tag{C.2}$$

under which the component fields transform as follows:

$$\phi \to \phi + \alpha_i X^i\,, \qquad \bar\phi \to \bar\phi + \alpha_i \bar X^i\,, \qquad \psi_\pm \to \psi_\pm + \alpha_i \frac{\partial X^i}{\partial \phi}\psi_\pm\,, \qquad \bar\psi_\pm \to \bar\psi_\pm + \alpha_i \frac{\partial \bar X^i}{\partial \bar\phi}\bar\psi_\pm\,, \tag{C.3}$$

where $\alpha^i, i = 1,2,3$ are real parameters. As eqs. (C.2), (C.3) show, only one of the isometries, generated by the $X^3$ Killing vector (an SO(2) $\subset$ SO(3) rotation), acts linearly:

$$\phi \to e^{-i\alpha_3}\phi, \psi_\pm \to e^{-i\alpha_3}\psi_\pm\,, \qquad \bar\phi \to e^{i\alpha_3}\bar\phi\,, \qquad \bar\psi_\pm \to e^{i\alpha_3}\bar\psi_\pm\,. \tag{C.4}$$

In fact, being the only linearly acting isometry, (C.4) is the only global symmetry exactly preserved on the lattice — the SO(3)/ SO(2) isometries fail, just like supersymmetry of the interaction terms, because of the failure of the Leibnitz rule for lattice derivatives, and are only preserved up to powers of the lattice spacing.[16]

If we now eliminate the auxiliary field $F$ from eq. (5.11), via its equation of motion, we arrive at a $Q_-, \bar Q_+$ invariant partition function with measure:

$$\prod_{\vec m} d\phi d\bar\phi\, (1+\phi\bar\phi)^2\, d\psi_+ d\bar\psi_+ d\psi_- d\bar\psi_-\,. \tag{C.5}$$

It is important that the measure (C.5) is invariant under the entire SO(3) isometry group (with action given in (C.3), which implies that $d\psi_\pm \to d\psi_\pm/(1+\alpha_i \partial X^i/\partial\phi)$, etc.); the presence of the $(1+\phi\bar\phi)^2$ factor in the measure is crucial to ensure this invariance. The "predecessor" of (C.5), eq. (5.12), is invariant under general local holomorphic field redefinitions. Note also that the seemingly divergent integral over the zero modes of $\phi, \bar\phi$ due to the measure factor in (C.5) is compensated by the curvature factor, $\sim (1+\bar\phi\phi)^{-4}$, coming from the fermion zero mode integration. Thus, the model with action (5.11), with $F$ eliminated, and measure given by (C.5), appears to define a finite partition function. Nevertheless, as explained in the end of section 5.2, it can not have the desired continuum limit: an examination of the perturbative spectrum of (5.11) reveals the presence of doublers, which ultimately lead to the vanishing of the $\mathrm{U}(1)_A$ anomaly.

In this section, we will attempt to lift the doublers by employing a lattice version of a known supersymmetric deformation of the continuum theory — the addition of twisted mass terms; this method has previously been applied in [17]. It amounts to first gauging a holomorphic isometry by introducing a corresponding vector superfield $V$, then giving an expectation value to the scalars in the vector multiplet, and subsequently decoupling

[15] See, for example [31].

[16] If one uses a real three vector to describe the bosonic sector of the model, the SO(3) global symmetry can be exactly preserved on the lattice; however, lattice supersymmetry is explicitly broken; see [30] (we note that fermion doubling and anomalies were not considered there).

all its fluctuations (this is described in superspace in [32]). The vevs of the scalars in $V$ lead to the appearance of a central charge in the supersymmetry algebra. As discussed above, on the lattice the only exact isometry we have at our disposal is the one generated by $X^3$ (C.4). We can describe the introduction of a twisted mass term in our superspace formalism by means of a spurion field $\mathcal{V} = \bar{\theta}^+\theta^- iz$, with $z$ assumed to be a real constant (so that $I$ maps $\mathcal{V} \to \mathcal{V}$), into (5.9), restricted to the $CP^1$ case:

$$S^{(1)}_{\rm twisted} = a^2 \sum_{\vec{m}} \int d\bar{\theta}^+ d\theta^- \frac{\bar{\Xi}_{\vec{m}}\, e^{\mathcal{V}}\, \Xi_{\vec{m}}}{(1+\bar{U}_{\vec{m}}\, e^{\mathcal{V}}\, U_{\vec{m}})^2}\,. \tag{C.6}$$

The introduction of the explicit $\bar{\theta}^+\theta^-$ into the superspace action breaks the $\theta$-shift symmetry; however, supersymmetry can be redefined by supplementing shifts of $\theta$ by appropriate field transformations. More precisely, the $\hat{\delta}$ variation (B.2) of $\mathcal{V}$ is $\hat{\delta} e^{\mathcal{V}} = iz\bar{\epsilon}^+\theta^- + iz\bar{\theta}^+\epsilon^-$. Thus, to restore "supersymmetry," we are forced to absorb $\hat{\delta} e^{\mathcal{V}}$ by redefining the action of supersymmetry, eq. (B.2), $\hat{\delta} \to \hat{\delta}'$ on the fields, in a manner consistent with the action of the involution:

$$\begin{aligned} \hat{\delta}' U &= \hat{\delta} U - iz\bar{\epsilon}^+\theta^- U\,, \qquad & \hat{\delta}'\Xi &= \hat{\delta}\Xi - iz\bar{\epsilon}^+\theta^-\Xi\,, \\ \hat{\delta}'\bar{U} &= \hat{\delta}\bar{U} - iz\bar{\theta}^+\epsilon^-\bar{U}\,, \qquad & \hat{\delta}'\bar{\Xi} &= \hat{\delta}\bar{\Xi} - iz\bar{\theta}^+\epsilon^-\bar{\Xi}\,. \end{aligned} \tag{C.7}$$

The corresponding modification of the component field transformations can be immediately read off from (C.7). In particular, one observes that only $\psi_-$, $\bar{\psi}_+$, $F$, and $\bar{F}$ experience an extra $z$- and $\epsilon$- dependent shift and that the measure (5.12) is invariant. It is straightforward to compute the anticommutator of the modified supersymmetry acting on the component fields and verify that (C.7) realizes the algebra with central charge, $\{\bar{Q}_+, Q_-\} = 2iz$.

The effect of introducing the spurion into (C.6) is easy to read off; for example, at the bilinear level, we see that (C.6) gives a U$(1)_A$ violating mass term $\sim z\bar{\psi}_-\psi_+$ to the fermions, but not to the scalars. Since introducing $\mathcal{V} = \bar{\theta}^+\theta^- iz$ breaks U$(1)_A$ but preserves all other symmetries, we have to revisit our arguments that lead to establishing eq. (5.9) as the most general lattice action consistent with the symmetries. The lattice supersymmetry as well as dimensional arguments (recall that $U$ is dimensionless, while $z$ has mass dimension 1), now allow general U$(1)_A$-violating, but U$(1)_V$, $I$, and $Z_4$ preserving terms of the form:

$$S^{(2)}_{\rm twisted} = -iza^2 \sum_{\vec{m}} \int d\bar{\theta}^+ d\theta^-\, f(\bar{U}_{\vec{m}} e^{\mathcal{V}} U_{\vec{m}})\,. \tag{C.8}$$

For the $CP^1$ model, the choice $f(x) = \frac{x}{(1+x)^2}$ yields, in the naive continuum limit, the continuum theory result for an SO(2) twisted mass term. In addition to the $\sim z\bar{\psi}_-\psi_+$ mass term from (C.6) one finds that (C.8) gives rise to additional fermion $\sim z\bar{\psi}_+\psi_-$ as well as scalar $\sim z^2\bar{\phi}\phi$ mass terms. The rest of the twisted model lagrangian can be easily worked out by expanding $S^{(1)}_{\rm twisted} + S^{(2)}_{\rm twisted}$ in components (with the $f$ given just below (C.8)). The naive continuum limit of the component lattice action is easily seen to coincide with the general formulae of [32].

Let us now ask whether twisted mass terms give an opportunity to lift the doublers while preserving the exact lattice supersymmetry. We start by introducing nonlocal twisted mass terms by inserting a lattice laplacian in the numerators of (C.6), (C.8):

$$S_r = a^2 \sum_{\vec{m}} \int d\bar{\theta}^+ d\theta^- r a^2 \frac{\bar{\Xi}_{\vec{m}} e^{\mathcal{V}} \Delta^2 \Xi_{\vec{m}} + \Delta^2 \bar{\Xi}_{\vec{m}} e^{\mathcal{V}} \Xi_{\vec{m}} - iz\, \bar{U}_{\vec{m}} e^{\mathcal{V}} \Delta^2 U_{\vec{m}} - iz\, \Delta^2 \bar{U}_{\vec{m}} e^{\mathcal{V}} U_{\vec{m}}}{(1 + \bar{U}_{\vec{m}} e^{\mathcal{V}} U_{\vec{m}})^2} . \tag{C.9}$$

When added to the original action, (C.1), it is easy to check that (C.9), upon taking the twisted nonlocal mass to scale (on dimensional grounds) as $z \sim 1/a$, gives mass of order $rz \sim r/a$ to the fermion doublers and of order $\sqrt{|r|}z \sim \sqrt{|r|}/a$ to the bosonic doublers. The different scaling with respect to $r$ for bosons and fermions already follows from the linearity of $S_r$ in $r$ and its being a mass term for fermions and mass squared term for the scalars (the auxiliary field is irrelevant for the scalar mass, in contrast with the case of superpotential Wilson term of section 5.3). Besides removing the doublers, the "Wilson action" $S_r$ violates the $\mathrm{U}(1)_A$ symmetry of (C.1); one expects, then, that the $\mathrm{U}(1)_A$ anomaly will be correctly reproduced in the continuum limit, as in the last reference in [24].

The drawback of using the nonlocal twisted mass terms (C.9) to eliminate the doublers is the explicit breaking of the lattice supersymmetry (as witnessed already by the different masses of the bosonic and fermionic doublers quoted above). The two terms in the action, (C.1) and (C.9), respect different supersymmetries: the original one (B.2) and the modified (C.7), respectively.

One could attempt to reconcile the two supersymmetries by also modifying the leading term by including an $e^{\mathcal{V}}$ factor there, so that both terms now respect the same twisted supersymmetry, i.e., by considering the total action:

$$S = S^{(1)}_{\text{twisted}} + S^{(2)}_{\text{twisted}} + S_r . \tag{C.10}$$

The problem with this approach is that while the twisted lattice supersymmetry is respected, the leading term gives an additional twisted local mass $\sim z$. Then, a study of the dispersion relation shows that there is no sensible limit of parameters allowing the doublers to decouple.

To summarize this section: it is possible to use twisted nonlocal mass term deformations of the nonlinear sigma model lattice action to remove the doublers and break the anomalous axial symmetry. However, the lattice supersymmetry is then explicitly broken. It is not immediately obvious whether the supersymmetric continuum limit can be achieved without fine tuning. A generic 2d $(2,2)$ nonlinear sigma model is a renormalizable theory (rather then superrenormalizable, or even finite, as is the LG model) and logarithmic divergences are present at any order of perturbation theory. Whether there exist models where the desired continuum limit (using the twisted nonlocal mass terms) can be achieved without fine tuning deserves further study.

## D. Power counting and finiteness of the supersymmetric lattice LG model

Here we consider naive power-counting to determine the superficially divergent subdiagrams that occur in lattice perturbation theory. In this exercise, we follow Reisz [26].

As alluded to in the main text, we can incorporate the Wilson terms of (5.17) directly into the superpotential action (5.14). This is done by a modification of the superpotential (5.18) as follows (note that we restrict to the simplest sort of superpotential interaction — cubic — and that some rescaling of coefficients has been performed):

$$W(U) = \sum_{\vec{m}} \left[\frac{m}{2} U_{\vec{m}}^2 + \frac{g}{3!} U_{\vec{m}}^3\right] - \frac{ra}{4} \sum_{\vec{m},\vec{n}} U_{\vec{m}} \Delta_{\vec{m},\vec{n}}^2 U_{\vec{n}} \,. \tag{D.1}$$

Then one defines $W'_{\vec{m}} = \partial W/\partial U_{\vec{m}}$ and $W''_{\vec{m},\vec{n}} = \partial^2 W/\partial U_{\vec{m}} \partial U_{\vec{n}}$. This moves the $r$-dependent terms of (5.19) into the coefficients denoted there as $W_I$ and $W_{IJ}$.

Note that in this formula and all others in our power-counting considerations, the lattice spacing $a$ is included in the definition of finite difference operators. For example, the lattice laplacian is:

$$\Delta_{\vec{m},\vec{n}}^2 = \frac{1}{a^2} \sum_{\mu=1,2} \left(\delta_{\vec{m}+\hat{\mu},\vec{n}} + \delta_{\vec{m}-\hat{\mu},\vec{n}} - 2\delta_{\vec{m},\vec{n}}\right) . \tag{D.2}$$

Following the construction outlined in section 5, we obtain the action:

$$\begin{aligned} a^{-2} S = & -i\bar{\psi}_- \Delta_{\bar{z}} \psi_- + i\psi_+ \Delta_z \bar{\psi}_+ - \bar{\phi} \Delta_z \Delta_{\bar{z}} \phi - \bar{F} F - \\ & -\bar{\psi}_- \overline{W}'' \bar{\psi}_+ - \psi_+ W'' \psi_- + W'(\phi)(F + i\Delta_{\bar{z}} \phi) + \overline{W}'(\bar{\phi})(\bar{F} - i\Delta_z \bar{\phi}) \,, \end{aligned} \tag{D.3}$$

Integrating out $F, \bar{F}$, the action becomes:

$$a^{-2} S = [\Delta_z \bar{\phi} + iW'(\phi)]_{\vec{m}} [\Delta_{\bar{z}} \phi - i\overline{W}'(\bar{\phi})]_{\vec{m}} + \bar{\chi}_{\vec{m}} M_{\vec{m},\vec{n}}(\phi, \bar{\phi}) \chi_{\vec{n}} \,, \tag{D.4}$$

where the fermions have been organized according to:

$$\bar{\chi} = (\psi_+, \bar{\psi}_-) \,, \qquad \chi = \begin{pmatrix} \psi_- \\ \bar{\psi}_+ \end{pmatrix} , \qquad M = \begin{pmatrix} -W''(\phi) & i\Delta_z \\ -i\Delta_{\bar{z}} & -\overline{W}''(\bar{\phi}) \end{pmatrix} . \tag{D.5}$$

The action can be written as $S = S_0 + S_{\rm int}$, where $S_0$ is quadratic in fields. Explicitly,

$$\begin{aligned} a^{-2} S_0 &= \bar{\phi} \left[-\Delta_z \Delta_{\bar{z}} + (\bar{m} - \frac{\bar{r}a}{2}\Delta^2)(m - \frac{ra}{2}\Delta^2)\right] \phi + \bar{\chi} M_0 \chi \,, \\ M_0 &= \begin{pmatrix} -(m - \frac{ra}{2}\Delta^2) & i\Delta_z \\ -i\Delta_{\bar{z}} & -(\bar{m} - \frac{\bar{r}a}{2}\Delta^2) \end{pmatrix} . \end{aligned} \tag{D.6}$$

For the interaction terms, it is convenient to introduce chirality projection operators for the fermions: $L = \frac{1}{2}(1 + \sigma_3)$ and $R = \frac{1}{2}(1 - \sigma_3)$. Then:

$$\begin{aligned} a^{-2} S_{\rm int} = & -g\phi\bar{\chi}L\chi - \bar{g}\bar{\phi}\bar{\chi}R\chi + \frac{1}{2}\bar{g}m\bar{\phi}^2\phi + \frac{1}{2}g\bar{m}\phi^2\bar{\phi} + \frac{1}{4}|g|^2\phi^2\bar{\phi}^2 - \\ & -\frac{1}{4}ra\bar{g}\bar{\phi}^2\Delta^2\phi - \frac{1}{4}\bar{r}ag\phi^2\Delta^2\bar{\phi} + \frac{i}{2}g\phi^2\Delta_{\bar{z}}\phi - \frac{i}{2}\bar{g}\bar{\phi}^2\Delta_z\bar{\phi} \,. \end{aligned} \tag{D.7}$$

Each loop integral has associated with it $\frac{1}{(Na)^2}\sum_{\vec{k}} \to \int \frac{d^2k}{(2\pi)^2}$. Since this scales like $a^{-2}$, the degree of divergence $D = 2$, just as in the continuum theory. $E_{B,F}$ are the number of external boson and fermion lines resp. $I_{B,F}$ count internal lines. The boson propagator is given by:

$$\tilde{G}_{\vec{k}} = \left[ a^{-2} \sum_{\mu} s^2(k_\mu) + |m(\vec{k})|^2 \right]^{-1} , \tag{D.8}$$

where here and below $s(k_\mu) = \sin(2\pi k_\mu/N)$, $c(k_\mu) = \cos(2\pi k_\mu/N)$ and

$$m(\vec{k}) = m + 2ra^{-1} \sum_{\mu} s^2 \left( \frac{k_\mu}{2} \right), \qquad m_s(\vec{k}) = \operatorname{Re} m(\vec{k}) \,, \qquad m_p(\vec{k}) = \operatorname{Im} m(\vec{k}) \,. \tag{D.9}$$

Since $\tilde{G}_{\vec{k}} \sim a^2$, it thus has $D = -2$, just as in the continuum. The Wilson mass term does not alter the degree of divergence, since what is important is the dependence on $a$. Similarly, the fermion propagator ($\gamma_3 = \sigma_3$, $\gamma_1 = -\sigma_2$, $\gamma_2 = \sigma_1$),

$$\tilde{D}_{\vec{k}} = \tilde{G}_{\vec{k}}^{-1} \left( m_s(\vec{k}) - i\gamma_3 m_p(\vec{k}) - ia^{-1} \sum_{\mu} s(k_\mu)\gamma_\mu \right) , \tag{D.10}$$

has $D = -1$, just as in the continuum.

The power counting for the vertices on the lattice, however, is different from the continuum. Below, $V_1$ is the number of $\phi^2\bar{\phi}$ vertices, $V_2$ is the number of $\phi^2\bar{\phi}^2$ vertices, $V_3$ is the number of $\phi\bar{\chi}L\chi$ vertices, and $V_4$ is the number of $\phi^2\Delta_{\bar{z}}\phi$ vertices. Conjugate vertices are also counted in these quantities. In contrast to the continuum, the vertices counted by $V_1$ and $V_4$ carry degree of divergence $D = 1$ because they scale like $a^{-1}$. For example, the $\phi\bar{\phi}^2$ vertices arise from:

$$S_{\text{int}} \ni a^2 \bar{g} \bar{\phi}^2 \left( \frac{1}{2} m\phi - \frac{1}{4} ra\Delta^2 \right) \phi \,. \tag{D.11}$$

Since $a\Delta^2 \sim a^{-1}$, the interaction associated with supersymmetrization of the Wilson mass term yields $D = 1$ for the corresponding vertex.

Taking these contributions to $D$ for a given diagram into account, as well as the usual constraints on lines and vertices, we obtain ($L$ is the number of loops):

$$\begin{aligned} D &= 2L - I_F - 2I_B + V_1 + V_4 \,, \\ L &= I_B + I_F - V_1 - V_2 - V_3 - V_4 + 1 \,, \\ E_B + 2I_B &= 3V_1 + 4V_2 + V_3 + 3V_4 \,, \\ E_F + 2I_F &= 2V_3 \,. \end{aligned} \tag{D.12}$$

We can eliminate $I_{B,F}$ and $L$ to obtain:

$$D = 2 - \frac{1}{2} E_F - V_1 - 2V_2 - V_3 - V_4 \,. \tag{D.13}$$

Taking into account constraints that arise from the Feynman rules, there exist 7 superficially divergent types of subdiagrams, consisting of tadpoles and 2-point functions; see figure 1.

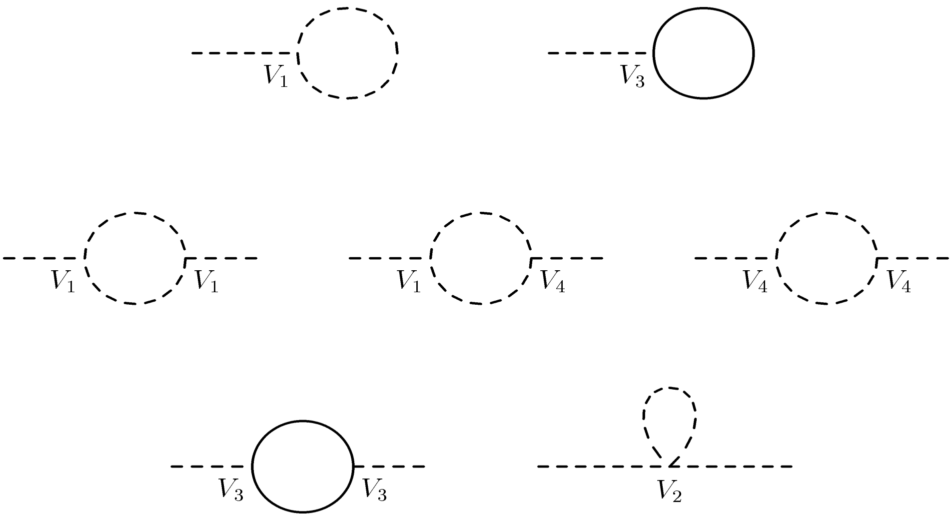


**Figure 1:** Types of superficially divergent diagrams. Dashed lines are bosons and solid lines are fermions.

A straightforward application of (D.7)–(D.10) shows that the net result for a particular choice of external lines either vanishes or is finite. In the case of the 1-point function, the two diagrams cancel exactly. In the case of the 2-point functions, $D = 0$. Hence it suffices to check at external momentum $\vec{k}_{ext} = 0$. This is because $d/d\vec{k} \sim a$, so that contributions at nonzero $\vec{k}$ are suppressed by further powers of $a$. The diagrams associated with $\langle\phi\phi\rangle$ cancel exactly at any $\vec{k}_{ext}$. The cancellations leading to these results are a consequence of the one exact supersymmetry.

The diagrams associated with $\langle\phi\bar{\phi}\rangle$ sum to a nonzero but finite quantity, thus deserving special mention. The $D = 0$ parts cancel exactly, again due to the exact lattice supersymmetry. What is left is only the $D = -2$ part. As shown by Reisz, $D < 0$ contributions to lattice perturbation theory necessarily approach their continuum values in the $a \to 0$ limit [26]. It follows that lattice perturbation theory using the $Q_A$-exact action is finite and reproduces the results of the continuum perturbation theory. This is an essential feature in rendering the $\mathcal{O}(a)$ irrelevant operators harmless in the $a \to 0$ limit.

Many of these cancellations arise due to the Ward identities that follow from the exact $Q_A$ supersymmetry, which also restricts finite renormalizations of the lattice action. For example, since $Q_A\psi_+ = F + i\Delta_{\bar{z}}\phi$, and since (as in quantum mechanics, see the end of appendix A) correlation functions of $Q$-exact quantities vanish, we find that

$$\langle F\rangle = -i\langle\Delta_{\bar{z}}\phi\rangle \tag{D.14}$$

exactly on the lattice. Translational symmetry then implies that the r.h.s. vanishes, hence $F$ tadpoles are forbidden (at the component loop level discussed above, this Ward identity implies that vanishing $\phi$ tadpoles imply vanishing of the $\phi^2$ graphs at zero momentum and vice versa).

We should also note that the consequences of the exact nilpotent $Q_A$ supersymmetry of the partition function transcend perturbation theory. The nilpotent supersymmetry of the partition function implies localization (see, e.g., the second reference in [23]). It should then be possible, through a careful application of localization, to study the Witten index for the LG models on the lattice, along the lines discussed in appendix A. We leave this for future work.

As far as the practical problems of simulations are concerned, at this point we also note that the fermion matrix (D.5) can be cast in a real basis, through the change of coordinates:[17]

$$\begin{aligned} \chi_1 &= \frac{1}{\sqrt{2}}(\eta_1 + i\eta_2)\,, \qquad & \chi_2 &= \frac{1}{\sqrt{2}}(\eta_1 - i\eta_2)\,, \\ \bar{\chi}_1 &= \frac{1}{\sqrt{2}}(\bar{\eta}_1 - i\bar{\eta}_2)\,, \qquad & \bar{\chi}_2 &= \frac{1}{\sqrt{2}}(\bar{\eta}_1 + i\bar{\eta}_2)\,. \end{aligned} \tag{D.15}$$

The matrix then becomes:

$$M = \begin{pmatrix} -\operatorname{Re} W'' + \Delta_2 & \operatorname{Im} W'' + \Delta_1 \\ -\operatorname{Im} W'' + \Delta_1 & -\operatorname{Re} W'' - \Delta_2 \end{pmatrix}. \tag{D.16}$$

For a square $N \times N$ lattice, $\det M > 0$, because the eigenvalues always come in complex conjugate pairs. However, for a more general $N_1 \times N_2$ lattice, we find that unpaired real eigenvalues can occur. These then allow for $\det M$ to be both positive or negative; however, in random gaussian samples of boson configurations we find that $\det M < 0$ occurs only rarely. For purposes of Monte Carlo simulation, the square lattices are much to be preferred, since they do not suffer from a sign problem.[18] These are the type of lattices studied in [15]. They allow for a faithful representation of the fermion determinant through real pseudofermions $y$:

$$S_{p.f.} = \frac{1}{2} y_{\vec{m}} [(M^T M)^{-1}]_{\vec{m},\vec{n}} y_{\vec{n}} \tag{D.17}$$

since $[\det(M^T M)]^{1/2} = \det M$ if $\det M$ is always positive. In particular, this allows for a hybrid Monte Carlo simulation, as was performed in [15]. In practice, this is an advantage since it avoids the systematic errors of, say, the $R$-algorithm.

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